# 拉格朗日相交理论的 Arnold 猜想

**学士学位论文**


作者　刘思阳

导师　胡建勋教授

学院　数学学院


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**摘要**

在这篇论文里面我们对拉格朗日相交理论的阿诺德猜想这一上世纪八十年代末的核心话题之一进行了一些探索。我们将主要专注于弗洛尔在他的一系列论文 [Flo88c, Flo88a, Flo88d, Flo89a, Flo89b] 中对阿诺德猜想在拓扑条件 $\pi_2(P, L) = 0$ 下的证明，并且在证明中的大多数部分使用后继者的一些略微推广。

在第一节里面我们对阿诺德猜想的历史作一个简要介绍，包括庞加莱的最后几何定理，阿诺德如何在他 1965 年的论文 [Arn65] 中提出这个猜想，以及他对这一问题的最初探索的简要概述。

接着我们展示弗洛尔对阿诺德猜想的证明方法。在第二节中，我们应用功泛函的变分问题来导出伪全纯带上的柯西-黎曼方程。这个方程指示了这个功泛函对应的轨道。我们证明这样的轨道会像期望的那样趋于临界点。

第三节是这篇论文的主要部分。我们构造弗洛尔链复形并且证明这确实是一个链复形，因而我们可以取上同调。这一步需要更多来自于非线性分析以及椭圆型偏微分方程的技巧。一些技巧性的部分放在了附录。

最终在最后一节，我们证明得到的上同调跟我们在构造中所进行的一般性选取无关，因而是一个拓扑不变量。接着我们能够将问题约化到“经典”相空间这一最简单的情形，并且证明弗洛尔上同调群跟莫尔斯上同调群同构，因此证明了阿诺德猜想。



**关键词：** 阿诺德猜想，莫尔斯理论，辛几何



**Abstract**

In this thesis we give an exposition of Arnold's conjecture on Lagrangian intersections, which was one of the main topics in late 1980s. We will mainly focus on Floer's proof under the topological condition $\pi_2(P, L) = 0$, given in his series of papers [Flo88c, Flo88a, Flo88d, Flo89a, Flo89b], with some slight generalizations made by precessors in most of the parts of the proof.

In the first section we give a brief introduction to the history of Arnold conjecture, including its origin: the Poincaré's last geometric theorem, how it was proposed by Arnold in his 1965 paper [Arn65], and a summary of his first exposition on this problem.

Then we present Floer's approach to this Arnold conjecture. In the second section we apply variation to an action functional and derive the Cauchy-Riemann equation on a pseudo-holomorphic strip, which indicates the trajectories associated to this action functional. We prove that this trajectories tend to critical points as expected.

The third section is the main part of this paper. We construct the Floer chain complex and prove that this is exactly a chain complex so that we could take the cohomology. This step requires much techniques from non-linear analysis and some study of non-linear elliptic partial differential equations. Some of the technical part is presented in the appendix.

Finally in the last section, we show that the given cohomology is independent of the choice of some generic structures along the process we construct the chain complex, so is actually a topological invariant. We could then reduce to the easiest case of a "classical" phase space and prove that the Floer cohomology group is isomorphic to the Morse cohomology group, hence proving the Arnold conjecture.

**Key words:** Arnold conjecture, Morse theory, symplectic geometry

# Contents

# 1. Introduction

This is an expository survery on the origin and the attempt people made to prove the Arnold conjecture, and we will focus especially on Floer's infinte-dimensional Morse theory on Lagrangian intersection and his proof of the Arnold conjecture in some restricted cases, with further developments to try to remove the restrictions given by Floer and prove Arnold conjectures for much more kinds of symplectic manifolds.

Arnold's conjecture comes originally from Poincaré's paper [Poi12] where he was concerning the restricted three-body problem in celestial mechanics and formalised the following problem, which was known as ¨ Poincaré's last geometric theorem":

**1.1 Theorem.** Let $A = \mathbb{S}^1 \times I$ be the closed annulus where $I = [0, 1]$ is the closed unit inteval and let $f : A \to A$ be an area-preserving homeomorphism satisfying the twist condition, i.e. $f$ preserves the orientation of one of the circles and reverses the other, then it must rotate the outer circle counterclocwise. Then $f$ must have

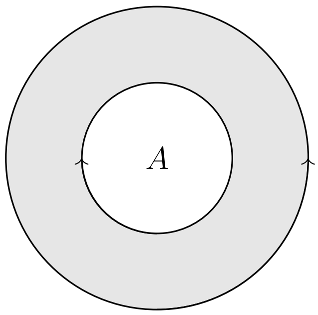


Figure 1: The Annulus and Twist Condition

at least two fixed points.

Although this was formalised by Poincaré, he failed to prove it in general case. One year after his paper was published, George Birkhoff gave a proof of this in [Bir13]. However, his proof could not be generalized to a general dimension, so many mathematicians are working to point out the generalization of this, including Arnold, who applies Morse theory to prove the Poincaré-Birkhoff theorem for $n$-torus in his short paper [Arn65]. He then proposed two general conjectures concerning fixed points, that is what we called the Arnold's conjecture.

**1a) Statement and Proof of Poincaré-Birkhoff Theorem.** Firstly let's review Arnold's proof of the original problem. In order to do this, we need some basic notions and facts about symplectic geometry.

**1.2 Definition.** We say a pair $(P^{2n}, \theta)$ is a **symplectic manifold** if $P$ is a differential manifold and $\theta$ is a differential 2-form on $P$ such that $\theta$ is closed and $\theta^n$ is nowhere vanishing. $\theta$ is called a **symplectic form**. If $(M, \sigma)$ is another symplectic manifold and $\varphi : P \to M$ is a smooth map, we say $\varphi$ is a **symplectomorphism** if $\varphi$ is a diffeomorphism and $\varphi^*\sigma = \omega$.

Symplectic manifolds was originally recognized as the phase space of a given mechanical system, see first few chapters of Arnold's book [Arn89] for examples, so the most natural example is the cotangent bundle $T^*Y$ of some differential manifold $Y$ of dimension $n$. The symplectic form $\theta$ on $T^*Y$ will be given by $\theta = -\mathrm{d}\lambda$ where $\lambda$ is a 1-form on $T^*Y$ such that $\lambda_{(y,\alpha)} = \alpha$. This is called the **tautological 1-form**. Another important example is the Riemann surface, where any area form will serve as a symplectic form and conversely. A more trivial example is the symplectic vector space $(V, \omega)$ where $V$ is odd-dimensional and $\omega$ is just a non-degenerate skew-symmetric bilinear form on $V$. In this case, we say a subspace $L \subset V$ **Lagrangian** if $\omega(x, y) = 0$ for all $x, y \in L$ and $L$ has its maximal dimension, i.e. $\dim L = n$.

**1.3 Definition.** A **Lagrangian submanifold** $L$ of $P$ is a submanifold of dimension $n$ such that $\theta|_L = 0$.

The symplectomorphism $\varphi$ will send a Lagrangian submanifold $L$ to a Lagrangian submanifold $\varphi(L)$.

**1.4 Example.** *Come back to the case of cotangent bundles, the zero-section $Y$ is a Lagrangian submanifold of $T^*Y$, and for any given smooth function $f\colon Y \to \mathbb{R}$, the graph of $\mathrm{d}f$ is a Lagrangian submanifold of $T^*Y$.*

The annulus $A$ in the Poincaré-Birkhoff theorem 1.1 can be regarded as a submanifold of the symplectic manifold $T^*\mathbb{S}^1$, written $A = \mathbb{S}^1 \times (0, 1)$, and in general the product manifold $\Omega = \mathbb{T}^n \times \mathbb{B}^n(1)$ in the cotangent bundle $T^*\mathbb{T}^n$(Note that since $\mathbb{T}^n$ is a Lie group, its cotangent bundle is trivial. See [Whi78].) In this case, we have the following result:

**1.5 Theorem** (Arnold)**.** Let $\varphi\colon \Omega \to \Omega$ be a symplectomorphism of $\Omega$ onto itself, and if $\varphi(\mathbb{T}^n)$ is a graph of some function $f \in C^\infty(\mathbb{T}^n)$, then the intersection of $\mathbb{T}^n$ and $\varphi(\mathbb{T}^n)$ has at least $2^n$ points(counted with multiplicity), where at least $n+1$ of them are geometrically different.

When $n = 1$, $\Omega = A$ and the intersection points of these two curves are just fixed points of the symplectomorphism $\varphi$, hence this is a slight restriction and a slight generlization of theorem 1.1. Arnold's proof involves the use of Morse theory.

**1.6 Definition.** Let $X$ be a general differential manifold and $f \in C^\infty(X)$ a smooth function on $X$. We say $f$ is a **Morse function** if all its singular points, i.e. $p \in X$ such that $\mathrm{d}f_p = 0$, are non-degenerate.

Non-degeneracy means that the second derivative of the function $f$ at the point $p$ is a non-degenerate bilinear form. If $p \in X$ is a singular point of $f$, then by Morse lemma

**1.7 Lemma** (Morse)**.** *Assume $f$ is a Morse function on $X$ and $p \in X$ a singular point of $f$, then there exists a neighbourhood $U$ of $p$ and a diffeomorphism $\varphi\colon \mathbb{B}^n(1) \to U$ such that $f \circ \varphi(x) = x_1^2 + x_2^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2$ for all $x \in \mathbb{B}^n(1)$.*

all the singular points of $f$ are isolated, hence if $X$ is compact, the number of singular points of $f$ is finite. For a proof, one can see Audin's book [AD14]. There's also a wonderful proof written by Nicolaescu, in his textbook [Nic11]. Moreover, we could define a chain complex $CM_*(X; f; \mathbb{F}_2)$ with coefficients in $\mathbb{F}_2$ generated by critical points $p$ of $f$, graded by the **Morse index**, denoted by $\mathrm{Ind}(p)$, which is the number of negative values of $\mathrm{d}^2 f_p$. The differential d of this chain complex will be the count of trajectories from one critical point to the other with Morse index one less. In order to achieve this, we consider the space $\mathcal{L}(x,y)$ of all integral curves of the gradient vector field of a Morse function $f$ connecting $x$ and $y$. If $f$ is Morse-Smale, i.e. for any two critical points $a,b\in \mathrm{Crit}(f)$ of $f$, the unstable manifold $W^u(a)$ of $a$ and the stable manifold $W^s(b)$ of $b$ intersect transversely, then by transversality we have $\dim(W^u(a)\cap W^s(b)) = \max\{\mathrm{Ind}(a)-\mathrm{Ind}(b),-1\}$. Let $\mathcal{M}(a,b)$ to be the moduli space of points $x\in X$ so that $x$ lies in one of the trajectories connecting $a$ and $b$, then we will have $\mathcal{L}(x,y)=\mathcal{M}(x,y)/\mathbb{R}$ where $\mathbb{R}$ is the translation by times, and hence $\dim\mathcal{L}(x,y)=\dim\mathcal{M}(x,y)-1=\mathrm{Ind}(x)-\mathrm{Ind}(y)-1$. Then if $\mathrm{Ind}(x)-\mathrm{Ind}(y)=1$ we have $\dim\mathcal{L}(x,y)=0$ and since it is a subset of the compact manifold $X$, we could count them. Once we define the differential $\partial$, we must verify that it satisfies $\partial^2=0$ so that this actually defines a complex. With much more effect, we could prove that the boundary of $\mathcal{L}(x,y)$ with $\mathrm{Ind}(x)-\mathrm{Ind}(y)=2$ consists of "brocken trajectories", which serves as the count of the coefficient of $y$ in $\partial^2 x$, and since we are working with $\mathbb{F}_2$ coefficient, it follows that $\partial^2=0$. Now we have defined the complex $(CM_*(X;f;\mathbb{F}_2),\partial)$, which is called the **Morse-Witten complex** since Witten in his paper [Wit82] firstly defined the differential in this complex, inspired from quantum field theory. Then we could prove that

**1.8 Proposition.** The homology $HM_*(X;f;\mathbb{F}_2)$ of the Morse-Witten complex, called **Morse homology** of $X$, is independent of the choice of $f$ and is isomorphic to the singular homology $H_*(X;\mathbb{F}_2)$.

The proof can be found in [AD14]. Let $b_i(X)$ be the $i$th Betti number of $X$, then from Proposition 1.8, we have

**1.9 Proposition** (Morse Inequality)**.** Assume $X^n$ is a compact differential manifold of dimension $n$, then we have the inequality

$$\#\,\mathrm{Crit}(f)\geq\sum_{i=1}^n b_i(X).$$

With Morse inequality, we could directly prove theorem 1.5.

*Proof of Theorem 1.5.* Note that $\Omega=\mathbb{T}^n\times\mathbb{B}^n(0;1)$, the tautological 1-form is given by $\lambda=p\mathrm{d}q$ where $(q,p)$ denotes the coordinates of $\Omega$. Consider the integral of $p\mathrm{d}q$ on $\varphi(\mathbb{T}^n)$

$$f(x)=\int_{x_0}^x p\mathrm{d}q,$$

where $x_0\in\varphi(\mathbb{T}^n)$ and $x\in\varphi(\mathbb{T}^n)$, then $f$ is well-defined since for any loop $\gamma$ on $\varphi(\mathbb{T}^n)$, we have

$$\int_\gamma p\mathrm{d}q=\int_{\varphi^{-1}(\gamma)}\varphi^*(p\mathrm{d}q)=\int_{\varphi^{-1}(\gamma)}p\mathrm{d}q=0$$

since $\varphi$ is a symplectomorphism. Then $f$ is differentiable on $\varphi(\mathbb{T}^n)$ with $\mathrm{d}f=p\mathrm{d}q$, hence the critical points of $f$ are the intersection of $\varphi(\mathbb{T}^n)$ with $\mathbb{T}^n$, hence if the intersection is transverse, the critical points will be non-degenerate, and therefore applying Morse inequality 1.9 we have $\#\,\mathrm{Crit}(f)\geq\sum_{i=0}^n b_i(\mathbb{T}^n)=2^n$. If the critical points are not non-degenerate, there are several cases to arise, one is that the intersection is infinite, and in this case the result follows directly. The second case is that two distinct critical points come together, but in this case we could apply the Lusternik-Schnilman theory to deduce that the lower bound for geometrically different critical points must be $n+1$. See [Nic11]. $\square$

**1b) The Arnold Conjecture.** Theorem 1.5 inspires Arnold in his paper [Arn65] to propose the following question: if we subtract the condition that $\varphi(\mathbb{T}^n)$ from theorem 1.5, will the same result hold? He then posted this questions in the book [Bro76] and was then known by American mathematicians. The Arnold conjecture can be stated formally as follows:

**1.10 Conjecture** (Arnold)**.** Assume that $P$ is a compact symplectic manifold of dimension $2n$ and $L \subset P$ a Lagrangian submanifold. If $\phi\colon P \to P$ is an exact symplectic automorphism such that $\phi(L) \pitchfork L$, then we have the following inequality

$$\#\phi(L) \cap L \geq \sum_{i=0}^{n} \dim b_i(L).$$

We say $\phi$ is a symplectic automorphism if it is a symplectomorphism from a symplectic manifold $(P, \theta)$ to itself keeping the symplectic structure $\theta$. We can form a infinite-dimensional Lie group $\operatorname{Symp}(P, \theta)$ of symplectic automorphisms of $(P, \theta)$ and $\operatorname{Symp}_0(P, \theta)$ the connected component of the identity. Now for all $\psi \in \operatorname{Symp}_0(P, \theta)$, there exists a symplectic isotopy $\{\psi_t\}_{0 \leq t \leq 1}$ such that $\psi_0 = \mathrm{id}$ and $\psi_1 = \psi$.

**1.11 Definition.** We say this isotopy is a **Hamiltonian isotopy** if there is a smooth function $H\colon P \to \mathbb{R}$ such that $\{\psi_t\}$ is the inverse flow of the Hamiltonian vector field $X_H$ of $H$.

A **Hamiltonian vector field** is a vector field $X_H$ associated to a smooth function $H\colon P \to \mathbb{R}$ such that $X_H \lrcorner \theta = -\mathrm{d}H$.

**1.12 Definition.** A symplectomorphism $\psi \in \operatorname{Symp}(P, \theta)$ is said to be **exact** if it is Hamiltonian isotopic to id.

With a given Hamiltonian vector field, we would obtain a slightly different conjecture from 1.10:

**1.13 Conjecture** (Arnold)**.** Assume $P$ is a compact symplectic manifold and $H\colon \mathbb{S}^1 \times P \to \mathbb{R}$ a periodic Hamiltonian, then the number of all periodic orbits of the Hamiltonian flow $\varphi_H^t$ is not less than the cup-length of $P$ plus one, and if all the periodic orbits are non-degenerate, then the number is not less than the sum of betti numbers of $P$.

Here the cup-length $CL(P)$ is defined by the maximal number of a set of differential 1-forms $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ such that $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k \neq 0$. From Lusternik-Schnilman theory we know that this is the minimum of the number of critical points of an arbitrary smooth functions over $P$. For a description of Lusternik-Schnilman theory, see [Nic11].

**1.14 Definition.** Assume that $\varphi_H^t$ is the flow of the Hamiltonian vector field $X_H$, then a period-1 point $x \in P$ is called **nondegenerate** if $\det(\mathrm{id}_{T_xP} - (\mathrm{d}\varphi_H^1)_x) \neq 0$.

The proof of this two conjectures go a quite long story. After Arnold's introducing this conjecture to [Bro76], Conley and Zehnder in their paper [CZ83] gave a first attempt to prove the Arnold conjecture for the torus with standard symplectic structure. In 1985, Gromov introduced an invariant given by counting the number of pseudo-holomorphic curves in his paper [Gro85] and proved the symplectic non-squeezing theorem with many important byproducts, which was then used by Andreas Floer in 1988-1989 to prove Arnold conjecture with some strong restrictions. Floer's proof was inspired not only by Gromov, but by Witten's paper [Wit82] which he used in infinite-dimensional case to construct the Floer complex and the Floer differential, and by his own paper [Flo88b] where he introduced the homology group which was only relatively graded with difference given by the spectral flow. In this paper, the author would give a precise review of Floer's idea of the proof and its further developments used to prove Arnold's conjecture in a more general case. The main result of this paper is that

**1.15 Theorem** (Floer)**.** Consider the symplectic manifold $(P, \theta)$ with a given Lagrangian submanifold $L$ such that $\pi_2(P, L) = 0$. Let $\{\phi_t\}_{0\leq t\leq 1}$ be a Hamiltonian isotopy such that $\phi_1(L) \pitchfork L$, then there exists a generic choice of families of $\theta$-compactible almost complex structures $\boldsymbol{J} = \{J_t\}_{0\leq t\leq 1}$ such that $L$ is totally real with respect to $J_0$ and $\phi_1(L)$ is totally real with respect to $J_1$, so that we have the isomorphism of cohomologies

$$HF(P, L; \phi, \boldsymbol{J}; \mathbb{Z}_2) \cong H^*(L; \mathbb{Z}_2)$$

where $HF(P, L; \phi, \boldsymbol{J}; \mathbb{Z}_2)$ is a cohomology group called the **Floer cohomology group**, which would be defined in section **??**. This cohomology group is generated by all the intersection points in $L \cap \phi_1(L)$, hence we obtain the lower bound estimate

$$|L \cap \phi_1(L)| \geq \sum_{i=0}^{\infty} \dim H^i(L; \mathbb{Z}_2),$$

which proves the Arnold conjecture.

# 2. Variations of Action Functional

The proof starts with a variation method, which is used to construct the complex and the differential. Let $P$ be a compact symplectic manifold and $L_0, L_1$ two transversal Lagrangian submanifolds of $P$. Consider the space

$$\Omega(P; L_0, L_1) = \{\gamma \in C^\infty([0,1], P) | \gamma(0) \in L_0, \gamma(1) \in L_1\}$$

of all paths starting from $L_0$ and ends at $L_1$, then any intersection point $p \in L_1 \cap L_2$ corresponds to a constant path in $\Omega(P; L_0, L_1)$, still denoted by $p$. Let $\Omega_0(P; L_0, L_1)$ be the subspace of $\Omega(P; L_0, L_1)$ consisting only of connected components of intersection points, then for each $\gamma \in \Omega_0(P; L_0, L_1)$ one can associate a homotopy class $[\Gamma]$ of homotopies connecting $\gamma$ to some intersection point $p$. Here we say two such homotopies $\Gamma_1$ and $\Gamma_2$ are homotopic if, we set the second variable $t$ to be curves from $L_0$ to $L_1$, $\Gamma_1(0,t) = \Gamma_2(0,t) \equiv p$, $\Gamma_1(1,t) = \Gamma_2(1,t) = \gamma(t)$, and $\Gamma_j(s,i) \in L_i$ for $j \in \{1,2\}$. This gives a fibre bundle $\widetilde{\Omega}_0(P; L_0, L_1)$ over $\Omega_0(P; L_0, L_1)$ and we can associate to any pair $(\gamma, [\Gamma])$ a value $\mathcal{A}(\gamma, [\Gamma]) = \int_\Gamma \Gamma^*\theta$.

**2.1 Lemma.** *This assignment is independent of the choice of the homotopy* $\Gamma$.

*Proof.* Assume $\Gamma_1, \Gamma_2$ are homotopic homotopies, and write $h\colon I^2 \times I \to P$ for this homotopy, then $h$ could descend to a continuous map $\bar{h}\colon K \to P$ where $K$ is the quotient space of $I^2 \times I$ such that $(0, t, u) \sim (0, 0, 0)$ and $(1, t, u) \sim (1, t, 0)$ for any $0 \leq t, u \leq 1$. The space $K$ would look like And we write $K_0$ for the

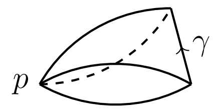


Figure 2: THE SPACE $K$

part of $\partial K$ lying in $L_0$, $K_1$ for the part lying in $L_1$, and $K_2$ for the upper part of $\partial K$ connecting $p$ to $\gamma$, and $K_3$ the lower part. Since we have $K_2$ and $K_3$ are homotopies with opposite orientations relative to $K$, by Stokes' formula,

$$\int_{I^2} \Gamma_1^* \omega - \int_{I^2} \Gamma_2^* \omega = \int_K h^* \mathrm{d}\omega - \int_{K_0} (h|_{K_0})^* \omega + \int_{K_1} (h|_{K_1})^* \omega = 0,$$

and therefore the action functional $\mathcal{A}$ is independent of the choice of the homotopy $\Gamma$. $\square$

It's obvious that the $C^\infty$-limit of any sequence of homotopies $\Gamma_i$ is again a homotopy $\Gamma$ that give the same homotopy class $[\Gamma]$, and for any given homotopy $\Gamma$, there exists a $C^\infty$ neighbourhood such that in this neighbourhood any homotopy represents the same homotopy class as $\Gamma$, hence the quotient set is discrete and the space $\widetilde{\Omega}_0(P; L_0, L_1)$ is a covering over $\Omega_0(P; L_0, L_1)$.

**2a) Gradient Flow Lines.** Now we apply the calculus of variations to compute the gradient vector field of the action functional $\mathcal{A}$ and compute the integral curve of the gradient vector field $\nabla \mathcal{A}$. Before this, let's review some basic notions about almost complex manifolds.

**2.2 Definition.** Assume $X^{2n}$ is a differential manifold of dimension $2n$. A section $J \in \Gamma(X; \mathrm{End}(TX, TX))$ is called an **almost complex structure** if $J^2 = -\operatorname{id}_{TX}$. Now if $(X^{2n}, \theta)$ is a symplectic manifold, we say $J$ is $\theta$**-compactible** if $g(-,-) = \theta(-, J-)$ gives a metric on $X$.

We call a differential manifold $(X, J)$ endowed with such an almost complex structure $J$ an **almost complex manifold**. It's an easy linear algebra exercise to see that

**2.3 Proposition.** For any symplectic manifold $(X, \theta)$, the space $\mathcal{J}(X, \theta)$ of almost complex structures on $X$ that is $\theta$-compactible is non-empty and contractible.

This result originally appeared in Gromov's paper [Gro85] and is contained in many textbooks and lecture notes, for example Mcduff & Salamon's famous book [MS17] and [AL94]. Here we present a proof.

*Proof.* For each $p$, the space $\mathcal{J}_p(X, \theta)$ of all almost complex structures on the tangent space $T_x X$ is identified with $\mathcal{J}(\mathbb{R}^{2n}; \omega_0)$, the set of all $\omega_0$-compactible complex structures on the symplectic vector space $\mathbb{R}^{2n}$. In the linear case, note that this set is homeomorphic to the homogeneous space $\mathrm{Sp}(\mathbb{R}^{2n}, \omega_0)/\mathrm{U}(n)$, which is contractible because of polar decomposition(for the statement of polar decomposition, see, for example, [Hal15] ). Then the conclusion follows from the fact that $J$ is a section of the bundle $\mathcal{J}_*(X, \theta) \to X$ with fibres $\mathcal{J}_x(X, \theta)$ at $x \in X$. $\square$

Now fix a point $p \in L_0 \cap L_1$ such that $\gamma$ is connected to $p$ in $\Omega(P; L_0, L_1)$, and since the set of equivalent classes of the homotopy $\Gamma$ is open and closed, for a sufficiently small perturbation of $\Gamma$, the homotopy class $[\Gamma]$ remains fixed. Now pick any variation $\alpha\colon I^2 \times (-\varepsilon, \varepsilon) \to P$ so that $\alpha(-, 0) = \Gamma$, and we compute the derivative of the composition $\mathcal{A} \circ \alpha(-, \lambda)$ with respect to $\lambda$ at $\lambda = 0$. To do this, we fix a $\theta$-compactible almost complex structure $J$ on $(X, \theta)$ and obtain a Riemannian metric $g$ on $X$.(Note that the almost complex structure given by a symplectic manifold is in fact an almost Kähler structure: we can use $g$ and $\theta$ to obtain a hermitian metric $h$ on $X$, which will give the hermitian connection on $X$.) Then we directly compute

$$\begin{aligned} \mathrm{d}\mathcal{A}_{(\gamma,[\Gamma])}\left(\frac{\partial \alpha}{\partial \lambda}\right) &= \int_{I^2} \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\theta(J\frac{\partial \Gamma}{\partial s}, J\frac{\partial \Gamma}{\partial t})\right) \mathrm{d}s\mathrm{d}t = \int_{I^2} \Gamma^*(\mathcal{L}_{\partial_\lambda \Gamma}\theta) = \int_{I^2} \Gamma^* \left(\mathrm{d}\frac{\partial \Gamma}{\partial \lambda} \lrcorner \theta\right) \\ &= \int_{\partial I^2} (\partial \Gamma)^* \left(\frac{\partial \Gamma}{\partial \lambda} \lrcorner \theta\right) = \int_0^1 \theta\left(\frac{\partial \Gamma}{\partial \lambda}, \dot{\gamma}\right) \mathrm{d}t. \end{aligned}$$

Therefore we could readily obtain two results, stated as follows:

**2.4 Proposition.** A pair $(\gamma, [\Gamma])$ is a critical point if and only if $\gamma$ is a constant loop, i.e. there exists a point $p \in L_0 \cap L_1$ such that $\gamma = p$.

The second result is that the gradient of $\mathcal{A}$ at the point $(\gamma, [\Gamma])$ is given by $\nabla \mathcal{A}_{(\gamma,[\Gamma])} = -J\frac{\partial \gamma}{\partial t}$, hence its integral curve $\alpha\colon [0, 1] \times \mathbb{R} \to P$ will satisfy the following partial differential equation

$$\frac{\partial \alpha}{\partial s} + J\frac{\partial \alpha}{\partial t} = 0, \tag{1}$$

which is called the **Cauchy-Riemann equation** with boundary conditions

$$\alpha(0, s) \in L_0,\ \forall s \in \mathbb{R}; \tag{2}$$

$$\alpha(1, s) \in L_1,\ \forall s \in \mathbb{R}. \tag{3}$$

By setting $\bar{\partial} = \frac{\partial}{\partial s} + J\frac{\partial}{\partial t}$, equation (1) can also be written of the form $\bar{\partial}\alpha = 0$. Hence gradient flow lines are just holomorphic strips $u\colon \mathbb{R} \times [0, 1] \to P$ satisfying certain boundary conditions.

**2.5 Remark.** *We also have a calculation of variation for the Hamiltonian case. In this case, there is a natural action functional* $\mathcal{L}(\gamma) = \int_{\mathbb{D}^2} \Gamma^*\theta - \int_0^1 H_t \circ \gamma \mathrm{d}t$ *where* $\gamma\colon \mathbb{S}^1 \to P$ *is a contractible loop and* $\Gamma$ *one homotopy from* $\gamma$ *to a point, which gives a map* $\mathbb{D}^2 \to P$*. In classical mechanics, this is just the usual action functional(Lagrangian). See for example, [Arn89]. The verification that the action functional is independent of the choice of homotopy relies on a simple fact that the symplectic form* $\theta$ *is exact near an isotropic submanifold. This result could be found in [MS17]. Then by a similar procedure, we could obtain the* **Floer equation**

$$\frac{\partial \alpha}{\partial s} + J\frac{\partial \alpha}{\partial t} + \operatorname{grad} H_t \circ \alpha = 0 \tag{4}$$

on a closed Riemann surface, i.e. without boundary conditions. Note that this is the usual Cauchy-Riemann equation with a perturbed term grad $H_t \circ \alpha$, and when $H \equiv 0$ we will have the usual Cauchy-Riemann equation. The proof of Arnold conjecture for periodic orbits relies on the analysis of this Floer equation. See [AD14].

**2b) Trajectories Tend to Critical Points.** Now we will prove a Morse-theoretic result that the constructed gradient flow lines $u\colon \mathbb{R}\times[0,1]$ must satisfy the property that for $s \to \pm\infty$, $u(\cdot,t) \to x\pm$ for some critical points $\{x\pm\} \subset L_0 \cap L_1$. Smale and Palais have been studied the condition for an infinite-dimensional Morse function to satisfy this property in their paper [PS64], but in this case the condition is slightly different: we are not directly proving that the action functional is actually a Morse function, but it satisfies the Palais-Smale condition mentioned in this paper, and hence we could convince ourselves that the trajectories(with bounded energy) will behave like the trajectories of a Morse function, which follows from an easy argument when we have proved that the Palais-Smale condition holds. In fact, we can also regard this as a corollary of Gromov compactness, that is, since the energy will tend to 0 when $s \to \infty$, the sequence of holomorphic strips given by translating in the $s$ direction will tend uniformly in compact subsets to a critical holomorphic strip(without bubbling). Floer proved this in a stronger requirement that $\pi_2(P, L_0) = 0$ in order to guarantee the bubbling phenomenon did not occur, but this was shown to be unnecessary via a deep study into the behaviour of the solution of the non-linear Cauchy-Riemann equation (1). This was first seen in Robbin and Salamon's paper [RS01] and the technical part was collected in Mcduff and Salamon's book [MS04]. Now we state the main result in this subsection.

**2.6 Theorem.** Let $u\colon \mathbb{R}\times[0,1]$ a smooth map with bounded energy satisfying PDE (1) with boundary conditions (2) and (3), then when $s \to \pm\infty$ we have

$$\lim_{s\to\pm\infty} u(s,\cdot) = x\pm,\ \ x\pm \in L_0 \cap L_1$$

and

$$\lim_{s\to\pm\infty} \|\frac{\partial u}{\partial s}\| = 0$$

where we pick an almost complex structure $J$ and the norm $\|\cdot\|$ is defined using the induced metric $g$.

Recall from Smale and Palais' paper that the trajectory of a Morse function on a Riemannian manifold will tend to critical points at infinity time if the following condition is satisfied:

**Palais-Smale condition** Let $(X, g)$ be a Riemannian manifold modeled on some Hilbert space $H$, $f\colon X \to \mathbb{R}$ a $C^2$-Morse function, then the Palais-Smale condition(or condition C) is that if a set $S$ of points in $X$ satisfies $|f(x)| \le M$ for $x \in S$ and $|\mathrm{d}f|$ is not bounded away from 0, then the closure of $S$ must contains a critical point of $f$.

This motivates our proof. We will need some results from the study of Cauchy-Riemann equations, given in appendix A. If the Palais-Smale condition is satisfied, it's easy to see that $\lim u(s,\cdot) = x\pm$ for some points $x\pm$: let $S$ be any subsequence

$\{u(s+s_k,\cdot)\}$ for a sequence $s_k \to \infty$, then from the energy bound the action functional has a bound on this set and the partial derivative tends to 0(which we will prove later), so there is a subsequence, still denoted $\{s_k\}$, that tends to some critical point $x+$. Now collect all the critical points on the space $\mathcal{M}(P; L_0, L_1; J)$ and consider the union $U$ of disjoint open neighbourhoods $U_p$ of them(since the critical points are disjoint and finite), and we claim that for sufficnetly large $k > 0$ we must have $u([s_k, +\infty)) \subset U_{x+}$. This follows from the fact that $U_{x+}$ is disjoint from the complement $U \setminus U_{x+}$ and that the image of $[s_k, +\infty)$ must be connected(when we view $\mathcal{M}$ as a topological space with the $C^\infty$-topology). $\square$

The rest of this paragraph is devoted to the verification of the Palais-Smale condition. Given a pseudo-holomorphic strip $u\colon \mathbb{R} \times [0,1] \to P$, we can define the **energy** of $u$ to be the functional

$$E(u) = \int_0^1 \int_{\mathbb{R}} \left\| \frac{\partial u}{\partial s} \right\|^2 \mathrm{d}s\mathrm{d}t$$

Since $u$ is pesudo-holomorphic, this is just the square of the norm of $\mathrm{d}u$. We set $\mathcal{M}(P; L_0, L_1; J) = \{u\colon \mathbb{R}\times[0,1] | u$ satisfies equation (1) with boundary conditions (2),(3), and $E(u) < \infty\}$ and $\mathcal{M}(x+, x-) = \{u | \lim\limits_{s\to\infty} u(s) = x+, \lim\limits_{s\to-\infty} u(s) = x-\}$. Now we write the Palais-Smale condition down explicitly in this case:

**2.7 Proposition.** Assume $u \in \mathcal{M}(P; L_0, L_1; J)$, then for any sequence $\{s_k\}_{k=1}^\infty$ that tends to $\infty$, there exists a subsequence $\{s_k'\}$ and a point $x+ \in L_0 \cap L_1$ such that

$$\lim_{k\to\infty} u_{s_k}(t) = \lim_{k\to\infty} u(s_k, t) = x+ \ \text{ in } C^\infty_{loc}$$

and

$$\lim_{s\to\infty} \left\| \frac{\partial u}{\partial s} \right\| \to 0.$$

Here the norm is taken with a chosen $\theta$-compactible almost complex structure $J$.

This is a very special case of what we call the "Gromov Compactness".

*Proof.* We want to apply the Arzela-Ascoli theorem to the sequence $\{u_k(s,t) = u(s_k+s, t)\}$, i.e. we want to verify that $\{u_k\}$ is uniformly bounded and is equicontinuous. The uniform boundedness follows from the fact that $P$ is compact, and the equicontinuity follows from the second result that we want to prove: that

$$\limsup_{k\to+\infty} \left| \frac{\partial u_k}{\partial s}(s,t) \right| \leq c$$

for some positive constant $c > 0$ and all $(s,t)$ in some compact subset $K \subset \mathbb{R} \times [0,1] := S$. This is the consequence of the mean-value inequality A.3 for Cauchy-Riemann equations: if this is not the case, then for $k$ sufficiently large, there must be a positive constant $\delta > 0$ such that $|\partial_s u_k(x_k, y_k)| \geq \delta$ for all $k$ and $(x_k, y_k) \in K$. Then by the mean-value inequality, we can pick $r$ sufficiently small such that $B_r(x_k, y_k) \cap S$ is disjoint, and the integration of $\partial_s u$ over $B_r(x_k, y_k)$ has a positive lower bound, contradicting the finiteness of the energy $E(u_k) = E(u)$. Therefore $u_k$ admits a subsequence, still written $\{u_k\}$, that converges locally in $C^0$ to some pseudo-holomorphic strip $u$. However, we also need the $C^1_{loc}$-convergence so that we can compute the energy of $u$. This relies on the elliptic regularity property, theorem A.8, which states that if $|\partial_s u_k|$ is bounded in some compact subset then the $C^2$-norm of $\{u_k\}$ in a smaller compact subset, hence Arzela-Ascoli theorem again tells us that $\{\partial_s u_k\}$ admits a $C^1_{loc}$-convergent subsequence, and if we calculate the energy of the limit $u$, we would find that $E(u) = 0$, hence $u$ would be a constant point $x+$, which is an intersection point of $L_0$ and $L_1$. Therefore the $C^1$-limit of $\{u_k\}$ consists of constant curves with energy 0, that is, the limit is a constant point $x+$. By a similar argument using regularity theorem A.8, we can also deduce that the $C^k_{loc}$-norm of $\partial_s u_k$ would also go to 0. □

**2.8 Remark.** *Furthermore, we can show that the $C^k$-norm of $\partial_s u$ is decreasing exponentially. The proof requires some effort on the analysis of the ordinary differential operator $\nabla_t + S$ where $S$ is the remaining term of the linearized Cauchy-Riemann operator. See [RS01] for details.*

**2.9 Remark.** *Although we do not try to prove the boundedness of the action functional, but with the Arzela-Ascoli theorem, it is easily seen that $\mathcal{A}(x+) - \mathcal{A}(x-) = E(u)$ for this pseudo-holomorphic strip $u$. Hence the Palais-Smale condition is satisfied.*

# 3. The Floer Complex

Here is the main part of the proof: we construct the Floer cochain complex $CF(P; L_0, L_1; J; \mathbb{Z}_2)$ with $\mathbb{Z}_2$-coefficients for a chosen almost complex structure $J$ and transverse Lagrangian submanifolds $L_0$ and $L_1$ of the symplectic manifold $(P, \theta)$. The definition is given as follows: we let

$$CF(P; L_0, L_1) = \bigoplus_{x \in L_0 \cap L_1} \mathbb{Z}_2 x$$

to be the vector space generated by intersection points, and to make it into a complex, we need a differential $\partial_F \colon CF(P; L_0, L_1) \to CF(P; L_0, L_1)$ such that $\partial_F^2 = 0$. Recall from Witten's Morse theory [Wit82] that the Morse differential $\partial_M$ of some Morse complex $CM(X; \mathbb{Z}_2)$ can be regarded as counting the number of trajectories from one critical point to another with Morse index one less. Here we also want to have a similar construction as in finite-dimensional case. In order to do this, we need to know the shape of the space $\mathcal{M}(x, y)$ of trajectories connecting two critical points $x$ and $y$. Recall from differential topology (See [Sma63], for example) that

**3.1 Definition.** Assume $X$ is a topological space. We say $X$ is a $C^k$**-Banach manifold** modeled over a Banach space $B$ if for all $x \in X$ there exists an open neighbourhood $U$ of $x$ such that $U$ is diffeomorphic to an open subset of $B$. It is callled a smooth Banach manifold if the diffeomorphism is $C^k$ for any $k \in \mathbb{Z}_{>0}$. Similarly we could also define the Hilbert manifold and the Frechét manifold.

The important result regarding the manifold structure of $\mathcal{M}(x, y)$ is the following property

**3.2 Definition.** Assume that $\Gamma \colon X \to Y$ is a $C^k$-map between $C^k$-Banach manifolds $X$ and $Y$, and $W$ is a regular submanifold of $Y$. If $\Gamma$ satisfies that for all $w \in W$ such that $w = \Gamma(x)$ for some $x \in X$, we have the composition

$$T_x X \xrightarrow{\mathrm{d}\Gamma_x} T_w Y \to T_w Y / T_w W$$

is a split surjection.

With the transversality property, it follows directly from definition(and implicit function theorem) that for each $y \in \mathrm{Im}\Gamma \cap W$, the preimage $\Gamma^{-1}(x)$ will be a $C^k$-Banach manifold. Moreover, recall the definition of Fredholm operator in the linear case:

**3.3 Definition.** Assume $f\colon X \to Y$ is a bounded linear operator between Banach spaces. We say $f$ is a **Fredholm operator** if $\mathrm{Im} f$ is a closed subspace of $Y$, $\ker f$ is finite-dimensional, and the quotient space $Y/\mathrm{Im} f$ is finite-dimensional.

In the non-linear case, we say a $C^k$-map $f\colon X \to Y$ between Banach manifolds is Fredholm if the tangent map $\mathrm{d}f_x$ is Fredholm for all $x \in X$, or in a more analytical setting, the linearization of $f$ at any $x \in X$ is Fredholm. If we have known $\Gamma$ is a Fredholm map, then it follows from Sard-Smale theorem [Sma65] that the set of regular values of $\Gamma$(i.e. the set of points $y \in Y$ such that the linearization $\mathrm{d}\Gamma_y$ is surjective) is residual, under the hypothesis that the Fredholm index of $f$ plus one, $\mathrm{Ind}(f) + 1$, is smaller than the order of smoothness of $\Gamma$. Then on each point $y \in Y$ with $\mathrm{d}\Gamma_y$ surjective, the preimage $\Gamma^{-1}(y)$ will be a $C^k$-Banach manifold of dimension exactly equal to the Fredholm index of $\mathrm{d}\Gamma_y$, therefore finite-dimensional.

**3a) Moduli Space as an Intersection** To apply the general theory to our case, we must give an alternative construction of $\mathcal{M}(x, y)$ such that it is an intersection of two known smooth Banach manifolds. Since $\mathcal{M}(x, y)$ is the set of all pesudo-holomorphic strips $u\colon \mathbb{R} \times [0, 1] \to P$ satisfying boundary conditions, that is, the set of smooth functions $u$ with $\bar{\partial} u = 0$, hence we could consider a larger set of functions from $\mathbb{R} \times [0, 1] := \mathbb{S}$ to $P$, with a Banach vector bundle associated to this function space, such that $\bar{\partial}$ can be viewed as a smooth section of this vector bundle, and $\mathcal{M}(x, y)$ as the intersection of the image of $\bar{\partial}$ with the zero-section of this vector bundle, and hence the transversality condition as well as the Fredholm property would be considered only on these intersection points. Note that the space of smooth maps even on compact manifolds does not form a Banach manifold, only Frechét manifolds, hence here we consider the function space given by weakly differentiable functions. Before the construction, let's prove an exponential decay property for maps in $\mathcal{M}(x, y)$ so that we could refine the Sobolev norm in our construction of function space so that all the functions in this manifold would decay exponentially at infinity:

**3.4 Theorem.** Assume $u \in \mathcal{M}(x, y)$, then there exists a sequence of positive constants $\{c_k\}_{k=1}^{\infty}$ and a positive constant $\varepsilon > 0$ such that

$$\|\partial_s u\|_{C^k([s,+\infty)\times[0,1]} \leq c_k e^{-\varepsilon s}.$$

Here the norm is taken with some fixed $\theta$-compactible almost complex structures. The proof of this requires some effort. From Proposition 2.6, we know that $u \in \mathcal{M}(x, y)$ tends as $s \to \infty$ to $y$ in the $C^1_{loc}$-topology, and by the elliptic regularity theorem A.8, we can argue by contradiction that $u$ tends to $y$ in the $C^\infty_{loc}$-topology as $s \to \infty$. If this is not the case, then there exists a sequence $\{(s_k, t_k)\}_{k=1}^{\infty}$ tending to $\infty$ such that $\|\partial_s u(s_k, t_k)\|_{C^k([-1,1]\times[0,1]} \geq \delta$ for some given positive constant $\delta > 0$. However, Arzela-Ascoli theorem tells us that this sequence has a convergent subsequence, and this convergent subsequence must tend to $y$, a contradiction. Then for any $u \in \mathcal{M}(x, y)$ and any neighbourhood $U$ of $y$, there is $M > 0$ such that for $s > M$, $u(s, t)$ lies in $U$ for all $t$. As in appendix 1, we let $J_0$ be the standard almost complex structure on $\mathbb{R}^{2n}$. With a little generality, in the following lemma we assume that we pick a family $\{J_t\}_{0 \le t \le 1}$ of almost complex structures on $P$ that is compactible with $\theta$, such that $L_0$ is totally real with respect to $J_0$ and $L_1$ totally real with respect to $L_1$. To distinguish between $J_t|_{t=0}$ and $J_0$, we use $J_0(q)$ for the almost complex structure $J_t|_{t=0}$ with value at $q$.

**3.5 Lemma.** *There exists a neighbourhood $U$ of $y$ and a local trivialization*

$$[0, 1] \times U \times \mathbb{R}^{2n} \to TP\colon (t, q, v) \mapsto \Phi_t(q)v \in T_qM$$

*such that*

1. $J_t(q)\Phi_t(q) = \Phi_t(q)J_0$*;*
2. $\Phi_t(q)(\mathbb{R}^n \times \{0\}) \subset T_qL_t$ *for* $t = 0, 1$*;*
3. $\theta_y(\Phi_t(y)v, \Phi_t(y)w) = \omega_0(v, w)$ *for* $v, w \in \mathbb{R}^{2n}$.

*Proof.* In $T_yP$, we have an induced symplectic form $\theta_y$. Let $\{\Lambda_t\}_{0 \le t \le 1} \subset LG(T_yP)$ be a family of Lagrangian submanifolds such that $\Lambda_i = T_yL_i$ for $i = 0, 1$. Then we could construct smooth functions $e_1(t), e_2(t), \cdots, e_n(t)\colon [0, 1] \to T_yP$ such that they form an orthogonal basis with respect to the metric $g_t = \theta_y(\cdot, J_t(y)\cdot)$.(This can be done by choosing a sequence of symplectic matrices $\{\Psi_t\}$ such that $\Psi_t(\Lambda_0) = \Lambda_t$, then for a given orthogonal basis $\{e_i(0)\}$ of $\Lambda_0$, we can set $e_i(t) = \Psi_t e_i(0)$.) Let $e_{i+n}(t) = J_t(y)e_i(t)$, and we obtain a smooth family of orthonormal basis of the tangent space $T_yP$. Now we could define the trivialization $\{\Phi_t\colon \mathbb{R}^{2n} \to T_yP\}$ by $\Phi_t v = \sum_{i=1}^{2n} v_i e_i(t)$ for $v \in \mathbb{R}^{2n}$. Then $\Phi_t$ identifies $\mathbb{R}^n \times \{0\}$ with $\Lambda_t$, $J_0$ with $J_t(y)$, and $\omega_0$ with $\theta_y$. We can then choose trivializations of $TP|_{L_0}$ and $TP|_{L_1}$ near $p$ such that $\mathbb{R}^n \times \{0\}$ is identified via trivializations to $T_pL_i$ when $t = i$. Finally, we could extend this trivialization to a trivialization over $U$ that identifies $J_0$ with $J_t$ for all $t$ and all $p \in U$. It is easy to see that the three conditions are already satisfied. □

The existence of such a good trivialization allows us to transform local vector fields near $y$ to smooth functions from $U$ to $\mathbb{R}^{2n}$, that is, for a large positive constant $M > 0$ and for all $(s, t)$ with $s > M$, set

$$\xi = \Phi_t(u(s, t))^{-1}(\frac{\partial u}{\partial s}),\ \eta = \Phi_t(u(s, t))^{-1}(\frac{\partial u}{\partial t}),$$

and let $\mathrm{D}_u$ be the linearization of the Cauchy-Riemann operator $\bar{\partial}$, that is, for any $\xi \in C^\infty(u^*TP)$,

$$\mathrm{D}_u(\xi) = \nabla_s\xi + J_u\nabla_t\xi + (\nabla_\xi J)\frac{\partial u}{\partial t} = \nabla_s\xi + \nabla_t\xi + S\xi$$

where $S$ is a matrix-valued smooth function on $\mathbb{S}$ and $\nabla$ is the Levi-Civita connection with respect to the Riemannian metric $u^*g := u^*(\theta(\cdot, J\cdot))$ on $u^*TP$. Now we transform the linearization $\mathrm{D}_u$ to the trivialization as follows: firstly, observe that in our case the Levi-Civita connection $\nabla_{s,t}$ is transformed into the usual partial derivative, denoted $\partial_{s,t}$, and we define the zero-order term $\tilde{S}$ as

$$\Phi_t(u)(\partial_s\xi + J_0\partial_t\xi + \tilde{S}\xi) = \mathrm{D}_u(\Phi_t(u)\xi)$$

it is also a matrix-valued function on $\mathbb{S}$. At $s = \infty$, the induced operator $S_\infty := \lim_{s\to+\infty} S$ is independent of $s$, hence we set

$$\Phi_t(y)\tilde{S}_\infty(t) := J_t(y)\partial_t\Phi_t(y)$$

to be the counterpart of $S_\infty$ in the trivialization. For convenience of notations, we omit the $\tilde{}$ from these operators and just write $S_\infty$ and $S$. We will expect that

**3.6 Proposition.** The matrix $S_\infty(t)$ is symmetric for every $t \in [0,1]$ and there exists a constant $c > 0$ such that

$$\|S(s,t) - S_\infty(t)\| \leq c(|\partial_s(s,t)| + d(u(s,t),y))$$

for all $s \geq 0$ and $t \in [0,1]$. Here $\|\cdot\|$ denotes the operator norm and $|\cdot|$ is just the vector norm with respect to the given Riemannian metric. Moreover, if $u$ satiafies a uniform $C^k$-bound for every positive integer $k \geq 1$, then there exists a positive constant $c_k > 0$ such that

$$\|S - S_\infty\|_{C^k([s,\infty)\times[0,1]} \leq c_k(\|\partial_s u\|_{C^k([s,\infty)\times[0,1]} + \sup_{s'\geq r, 0\leq t\leq 1} d(u(s',t),y)$$

for every $s \geq 0$. Here we assume that $u([0,\infty)\times[0,1]) \subset U$.

Here we only assume $u$ to be an arbitrary map from $[0,+\infty)\times[0,1]$ into $U$.

*Proof.* Assume that $u$ is the restriction of $u$ to the subspace $[0,\infty)\times[0,1]$. By a direct calculation, for any $v, w \in \mathbb{R}^{2n}$, we have

$$\begin{aligned} g_0(v, S_\infty(t)w) &= \omega_0(v, J_0 S_\infty(t)w) = \theta_y(\Phi_t(y)(v), J_t\Phi_t(y)S_\infty(t)w) = \theta_y(\Phi_t(y)(v), J_tJ_t\partial_t\Phi_t(y)w) \\ &= -\theta_y(\Phi_t(y)v, (\partial_t\Phi_t(y))w) = \theta_y(\partial_t\Phi_t(y)v, \Phi_t(y)w) = g_0(S_\infty(t)v, w), \end{aligned}$$

hence $S_\infty(t)$ is symmetric. Since when we take $v$ to be the constant vector on $[0,\infty)\times[0,1]$, we will have

$$\Phi_t(u)Sv = \nabla_s(\Phi_t(u)v) + J_t(u)\nabla_t(\Phi_t(u)v) + (\nabla_{\Phi_t(u)v}J_t(y))\frac{\partial u}{\partial t}$$

hence the difference can be computed as

$$\begin{aligned} (S(s,t) - S_\infty(t))v = \Phi_t^{-1}(u)\left(\nabla_s(\Phi_t(u)v) + J_t(u)\nabla_t(\Phi_t(u)v) + (\nabla_{\Phi_t(u)v}J_t(y))\frac{\partial u}{\partial t}\right) \\ - \Phi_t^{-1}(y)J_t(y)\partial_t\Phi_t(y) \quad (5) \end{aligned}$$

Since we have

$$\Phi_t^{-1}(u)(\nabla_{\Phi_t(u)v}J_t(y)) = J_t(y),$$

the norm of this term reduces to $\partial_s u$; for the other part, note that $\nabla_s(\Phi_t(y)v) = 0$, by mean-value inequality, this is controlled by the derivatives of $\Phi_t$ and the distance $d(u(s,t),y)$. Therefore we have proved the estimate for the operator norm. For the $C^k$-estimate, note that by taking derivatives of $S$ with respect to $s$, we have that

$$\begin{aligned}\Phi_t(u)(\partial_s Sv) &= \nabla_s(\Phi_t(u)Sv) - (\nabla_s\Phi_t(u))Sv \\ &= \nabla_s\nabla_s(\Phi_t(u)v) + (\nabla_s J_t(u))\nabla_t(\Phi_t(u)v) + J_t(u)\nabla_s\nabla_t(\Phi_t(u)v) \\ &+ \nabla_s(\nabla_{\Phi_t(u)v}J_t(u)\partial_t u) - (\nabla_s\Phi_t(u))Sv.\end{aligned}$$

Each term in the above expression corresponds to derivatives of $u$ or the distance $d(u,y)$, thus we have proved the required estimate for $k=1$ and by a direct induction process we obtain the estimate for any $k$. $\square$

It follows from the estimate that in our case, when $s\to\infty$ the operator $S(s,t)$ tends in $C^\infty$ topology to $S_\infty(t)$, a symmetric operator on $T_yP$. Now we try to prove the exponential decay for the $C^1$-topology. Consider the Hilbert space $H=L^2([0,1];\mathbb{R}^{2n})$ and the dense subspace

$$V=\{\xi\in W^{1,2}([0,1],\mathbb{R}^{2n})|\xi(0),\xi(1)\in\mathbb{R}^n\times\{0\}\},$$

and consider the two differential operators $A(s):=J_0\partial_t+\frac{1}{2}(S(s,t)+S(s,t)^T)$ and $A_\infty:=J_0\partial_t+S_\infty$. By this slight modification we know that $A(s)$ is also a self-adjoint operator on $H$. Now define $B(s)\colon H\to H$ by $B(s)=\frac{1}{2}(S(s,t)-S(s,t)^T)$ to be the skew-symmetric operator such that $A(s)+B(s)=J_0\partial_t+S$. There's a final preliminary lemma that gives a good estimate for the unbounded opeartor $A_\infty$.

**3.7 Lemma.** *The operators $A(s)-A_\infty$, $\dot{A}(s)$ and $B(s)$ extend to bounded operators on $H$, $A_\infty\colon V\to H$ is bijective, and we have the following estimate*

$$\|A(s)-A_\infty\|_{\mathcal{L}(H)}+\|B(s)\|_{\mathcal{L}(H)}\le c\sup_{0\le t\le 1}(|\partial_s u(s,t)|+d(u(s,t),y))$$

*and*

$$\|\dot{A}(s)\|_{\mathcal{L}(H)}\le c\sup_{0\le t\le 1}(|\nabla_s\partial_s u(s,t)|+|\partial_s u(s,t)|+d(u(s,t),y)).$$

*Proof.* These two estimates follows directly from the previous Proposition, so we only focus on the bijectivity of $A_\infty$. We first show that $A_\infty$ is injective. Given $\xi\in V$ such that $A_\infty=0$, then $\xi$ is a smooth path satisfying the boundary conditions $\Phi_0(y)\xi(0)\in T_yL_0$ and $\Phi_1(y)\xi(1)\in T_yL_1$, thus by a direct calculation,

$$\partial_t(\Phi_t(y)\xi(t))=\Phi_t(y)\partial_t\xi(t)-J_t(y)\Phi_t(y)S_\infty(t)\xi(t)=\Phi_t(y)(\partial_t-J_0S_\infty(t))\xi(t)=0$$

since $A_\infty=J_0\partial_t+S_\infty(t)$. Therefore $\Phi_t(y)\xi(t)$ is a constant path with $\Phi_t(y)\xi(t)\in T_yL_0\cap T_yL_1$, therefore $\Phi_t(y)\xi(t)=0$, i.e. $\xi(t)=0$ by transversality.

Now we show that $A_\infty$ is surjective. Note that $A_\infty$ is an ordinary differential operator on $V$, hence by the standard theory of ordinary differential equations, we obtain regardless of the boundary condition a fundamental solution

$$\Psi\colon[0,1]^2\to\mathrm{Sp}(2n)$$

satisfying $\Psi(t,t) = \mathrm{id}_{\mathbb{R}^{2n}}$ and $A_\infty\Psi(t,t') = 0$ for all $(t,t') \in [0,1]^2$. Now we insert the boundary condition and use the fact that $A_\infty$ is injective to obtain that for $\Lambda_0 = \mathbb{R}^n \times \{0\}$, the map

$$\Lambda_0 \times \Lambda_0 \to \mathbb{R}^{2n} \colon (\xi_0, \xi_1) \to \xi_1 - \Psi_\infty(1,0)\xi_0$$

must be bijective. This is because, when we take $\xi(t) = \Psi(t,0)\xi_0$, if the above map sends $(\xi_0,\xi_1)$ to 0 then we must have $\xi(1) \in \Lambda_0$, so $\xi \in V$ and by the condition of fundamental solution, we have $A_\infty\xi = 0$, therefore $\xi = 0$, and hence $\xi_0 = 0$ and $\xi_1 = \xi(1) = 0$. Therefore the map is injective. Surjectivity follows from the dimension of these two vector spaces. Now for $\eta \in H$, we construct the vector $\xi \in V$ with $A_\infty\xi = \eta$. Using bijectivity, we can find $(\xi_0,\xi_1) \in \Lambda_0 \times \Lambda_0$ such that

$$\xi_1 - \Psi(t,0)\xi_0 = -\int_0^1 J_0\Psi(1,t')\eta(t')\mathrm{d}t'$$

This is the boundary value of $\xi$, and we construct $\xi$ using variation of constants, i.e.

$$\xi(t) = \Psi(t,0)\xi_0 - \int_0^t J_0\Psi(t,t')\eta(t')\mathrm{d}t'. \qquad \square$$

Now the exponential decay follows from an abstract result concerning ODE in Hilbert spaces:

**3.8 Lemma.** *Assume that $\xi\colon [0,+\infty) \to H$ and $\eta\colon [0,+\infty) \to H$ are continuously differential functions such that $\xi(s) \in V$ for all $s \geq 0$ and*

$$\dot{\xi}(s) + A(s)\xi(s) + B(s)\xi(s) = \eta(s).$$

*We assume furthermore that $\eta$ satisfies the exponential decay property*

$$\|\dot{\eta}(s)\| + \|\eta(s)\| \leq ce^{-\varepsilon s}$$

*for some positive constants $c, \varepsilon > 0$, then there exists positive constants $c_1 > 0$ and $\delta > 0$ such that $\|\xi(s)\| \leq c_1 e^{-\delta s}$.*

*Proof.* Consider the function

$$u(s) = \frac{1}{2}\|\xi(s)\|^2$$

then by a direct calculation, we have $\dot{u}(s) = \langle \xi(s), \eta(s) - A(s)\xi(s) - B(s)\xi(s)\rangle = \langle \xi(s), \eta(s) - A(s)\xi(s)\rangle$ since $B(s)$ is anti-symmetric by definition. Taking second-order derivatives, we have

$$\begin{aligned}
\ddot{u}(s) &= \langle \dot{\xi}(s), \eta(s) - 2A\xi(s)\rangle + \langle \xi(s), \dot{\eta}(s) - \dot{A}\xi\rangle \\
&= \|\eta\|^2 + 2\|A\xi\|^2 - 3\langle A\xi, \eta\rangle - \langle B\xi, \eta\rangle + 2\langle B\xi, A\xi\rangle \\
&\geq \|\eta\|^2 + \|A\xi\|^2 - \left(\frac{5}{2}\|B\|^2 + \|\dot{A}\|^2\right)\|\xi\|^2 - 4\|\eta\|^2 - \langle \xi, \dot{\eta}\rangle.
\end{aligned}$$

By the fact that $A - A_\infty$, $\dot{A}$ and $B$ decrease to 0 when $s \to \infty$, and $\|\eta\|, \|\dot{\eta}\|$ satisfies the exponential decay property, and the fact that $A_\infty$ is bijectivve, we could find a positive constant $\delta < \min\{\varepsilon, 1/4\}$ such that

$$\|A\xi\| \geq 3\delta\|\xi\|$$

then there exists $s_0 \geq 0$ so that

$$\frac{5}{2}\|B(s)\|^2 + \|\dot{A}(s)\|^2 \leq \delta^2 \quad \text{and} \quad \|A(s) - A_\infty\| \leq \delta$$

for all $s \geq s_0$, thus we obtain the further estimate

$$\begin{aligned}\ddot{u}(s) &\geq 4\delta^2\|\xi\|^2 - \delta^2\|\xi\|^2 - 4\|\eta\|^2 - \langle \xi, \dot{\eta}\rangle \geq 3\delta^2\|\xi\|^2 - 4\|\eta\|^2 - \delta^2\|\xi\|^2 - \frac{1}{4\delta^2}\|\eta\|^2 \\ &\geq 2\delta^2\|\xi(s)\|^2 - \frac{1}{\delta^2}(\|\eta(s)\|^2 + \|\dot{\eta}(s)\|^2) \geq 2\delta^2\|\xi(s)\|^2 - \frac{c}{\delta^2}e^{-2\varepsilon s} = (2\delta)^2 u(s) - c_2 e^{-2\varepsilon s}.\end{aligned}$$

Now we introduce the auxillary function

$$\beta(s) = u(s) + \frac{c_2 e^{-2\varepsilon s}}{(2\varepsilon)^2 - (2\delta)^2}$$

so that by a direct calculatioon, we have

$$\ddot{\beta}(s) \geq (2\delta)^2\beta(s).$$

Now the conclusion will follow from a standard theory of ordinary differential equations. We first show that this would imply that

$$\dot{\beta}(s) + 2\delta\beta(s) \leq 0$$

For all $s \geq s_0$. If not, there is $s_1 \geq s_0$ such that $\dot{\beta}(s_1) + 2\delta\beta(s_1) > 0$, by taking derivatives we find that for all $s \geq s_1$, we must have this inequality, thus there exists a positive constant $C > 0$ such that

$$\dot{\beta}(s) + 2\delta\beta(s) \geq Ce^{2\delta s}$$

therefore we have

$$\beta(s) \geq Ce^{2\delta s} - D$$

for all $s \geq s_1$ and for some positive constants $C$ and $D$. However, from the fact that $\eta = \dot{\xi} + A\xi + B\xi$ and that $\eta$ satisfies the exponential decay, it follows that $\|\xi\|$ cannot diverge as $s \to \infty$. This gives a contradiction, therefore we have $\dot{\beta}(s) + 2\delta\beta(s) \leq 0$. Since $\beta(s)$ is positive, it follows that there is a positive constant $C > 0$ such that for all $s \geq s_0$, $\beta(s) \leq e^{-2\delta s}C$ and therefore $\|\xi(s)\| \leq \sqrt{2u(s)} \leq \sqrt{2\beta(s)} \leq Ce^{-\delta s}$. □

*Proof of Proposition 3.4.* What remains is that the estimate holds for the $C^k$-norm for all $k \geq 1$. Recall from elliptic regularity result A.8 that for every $k \geq 1$, there is a positive constant $c_k > 0$ such that for all $s \geq 1$ we have

$$\|\xi\|_{W^{k,2}([s,\infty)\times[0,1]} \leq c_k(\|S\xi\|_{W^{k-1,2}[s-1,\infty)\times[0,1]} + \|\xi\|_{W^{k-1,2}([s-1,\infty)\times[0,1]}) \leq c_k\|\xi\|_{W^{k-1,2}([s-1,\infty)\times[0,1])}$$

where $S$ is the order zero term in the linearized Cauchy-Riemann operator. Now by induction, for $s \geq k$ we have the exponential estimate for $\|\xi\|_{W^{k,2}([s,\infty)\times[0,1]}$ and therefore we have the exponential decay for all $C^k$-norm (as $k$ tends to $\infty$ and use the embedding theorem of Sobolev spaces). □

With the exponential decay at hand, we know that $\mathcal{M}(x,y) \subset C^{\infty}_{\searrow}(x,y)$ where the latter space denotes the space of all smooth strips with ends $x, y$, satisfying the boundary conditions, and converges exponentially in the $C^{\infty}$-topology. However, this is again not a Banach manifold, so we consider the Banach manifold locally modeled on Sobolev spaces that with the exponential decay property. For each $u \in C^{\infty}_{\searrow}(x,y)$, we consider firstly the tangent space $W^{1,p}(u^*TP)$ and for each $\xi \in W^{1,p}(u^*TP; L_0, L_1)$ =the set of all $W^{1,p}$-sections satisfying the boundary condition, we use the "exponential map" to construct a neighbourhood of the Banach manifold $\mathcal{P}^{1,p}(x,y)$ as the set

$$\{v | \exists \xi \in W^{1,p}(u^*TP), v(s,t) = \exp_{u(s,t)} \xi(s,t) \forall (s,t) \in \mathbb{S}\}$$

where we pick a compactible almost complex structure $J$ and exp is the exponential map on $P$ of the given Riemannian metric $g$. We then define a Banach bundle $\mathcal{L}^p(x,y) \to \mathcal{P}^{1,p}(x,y)$ as the set of pairs $(u,\xi)$ with $\xi \in L^p(u^*TP; L_0, L_1)$. Then $\bar{\partial}$ can be viewed as a section on this Banach bundle. It's a technical result of Floer in his paper [Flo88d] that

**3.9 Proposition.** The Banach manifolds and the Banach bundles constructed above are all smooth, and the section $\bar{\partial}$ is a smooth section on this Banach bundle.

Therefore it suffices for us to give the transversality property for the smooth section $\bar{\partial}$.

**3b) The Fredholm Property** Before we enter the transversality of $\bar{\partial}$, we give a proof that the linearized Cauchy-Riemann operator $\mathrm{d}\partial$ is Fredholm. This would be essential in our discussion of transversality property as well as the calculation of dimensions of the moduli space. The main result of this section is

**3.10 Theorem.** The linearized Cauchy-Riemann operator

$$L = \bar{\partial} + S(s,t) \colon W^{1,p}(\mathbb{S}, \mathbb{R}^{2n}) \to L^p(\mathbb{S}, \mathbb{R}^{2n})$$

is Fredholm for all $p > 1$, where we abbreviate $W^{1,p}(\mathbb{S}, \mathbb{R}^{2n}; L_0, L_1)$ for $W^{1,p}(\mathbb{S}, \mathbb{R}^{2n})$.

The proof of the Fredholm property relies on the fact that the dual operator of $L$, $L^*$, is also elliptic, and with the "semi-Fredholm" property

**3.11 Proposition.** Assume that $A \colon X \to Y$ is a bounded linear operator such that there exists a positive constant $C > 0$ and a compact operator $K \colon X \to Y$ such that for all $x \in X$,

$$\|x\|_X \leq C(\|Ax\|_Y + \|Kx\|_Y),$$

then $A$ has finite-dimensional kernel and a closed image.

This is not hard to prove, and can be found in [Bre11]. Since the dual operator satisfies a similar estimate, we have the kernel of $L^*$ is also finite-dimensional, but it is exactly the cokernel of $L$. Therefore this semi-Fredholm property would imply that $L$ is Fredholm. We obtain from the elliptic estimate for $L$ in proposition A.9 that there exists a positive constant $C > 0$ with

$$\|Y\|_{W^{1,p}(\mathbb{S})} \leq C(\|LY\|_{L^p(\mathbb{S})} + \|Y\|_{L^p(\mathbb{S})})$$

but $\mathbb{S}$ is non-compact and we cannot directly use Rellich compactness. But if we have

**3.12 Proposition.** The operator $L$ is bijective from $W^{1,p}(\mathbb{S}, \mathbb{R}^{2n})$ to $L^p(\mathbb{S}, \mathbb{R}^{2n})$ provided $S$ is independent of $s$.

then $L$ is asympototically bijective, so we could improve this inequality into

$$\|Y\|_{W^{1,p}(\mathbb{S})} \leq C(\|LY\|_{L^p(\mathbb{S})} + \|Y\|_{L^p([-M,M]\times[0,1])})$$

where $M > 0$ is a sufficiently large positive number. Then we can apply the Rellich compactness theorem to obtain that $W^{1,p}(\mathbb{S}, \mathbb{R}^{2n}) \to L^p([-M, M] \times [0, 1], \mathbb{R}^{2n})$ is a compact operator, proving the main theorem 3.10. □

Now we proceed to prove Proposition 3.12. With notation in the previous subsection, we can write $L = \nabla_s + A_\infty$. Lemma 3.7 gives the bijectiveness of $A_\infty$ in the case $p = 2$ and more generally for all $p \geq 2$(since in this case $\xi$ would be continuous and it makes sense to consider the boundary conditions), we know prove the bijectiveness of $L$ in the case $p = 2$ first.

**The case $p = 2$.** We are going to use Hille-Yosida theorem stated in chapter 7 of [Bre11], which states that

**3.13 Theorem** (Hille-Yosida)**.** If $A$ is a closed unbounded operator on a Hilbert space $X$ with dense domain and is maximally monotone, i.e. $\mathrm{id}_X + A$ is surjective, then the ordinary differential equation

$$\begin{cases} \frac{\partial Y}{\partial s} + AY = 0, \\ Y(0) = Y_0 \end{cases} \tag{6}$$

has a unique solution $Y\colon [0, +\infty) \to X$. Moreover, for all $s > 0$, we have the following estimate

$$\|Y(s)\| \leq \|Y_0\|, \quad \left\|\frac{\partial Y}{\partial s}\right\| = \|AY(s)\| \leq \frac{1}{s}\|Y_0\|.$$

However, $A_\infty$ is not monotone in $L^2([0, 1], \mathbb{R}^{2n})$. Recall that $A_\infty$ is bijective from $W^{1,2}([0, 1], \mathbb{R}^{2n})$ to $L^2([0, 1], \mathbb{R}^{2n})$, the inverse $A^{-1}$ is then continuous by the Banach-Steinhaus theorem. Moreover, the inclusion $W^{1,2}([0, 1], \mathbb{R}^{2n}) \to L^2([0, 1], \mathbb{R}^{2n})$ is compact, hence $A^{-1}$ is a compact operator. Observe also that $A$ is a self-adjoint operator, hence from the spectral theory for compact self-adjoint operators, $L^2([0, 1], \mathbb{R}^{2n})$ can be decomposed into the direct sum of eigenspaces of $A^{-1}$. Write $H = L^2([0, 1], \mathbb{R}^{2n})$ and decompose $H$ as

$$H = H^+ \oplus H^-$$

where $H^\pm$ is the subspace of $H$ consisting of direct sum of positive and negative eigenspaces of $A^{-1}$. Noticing that for $x$ an eigenvector we have $A^{-1}x = \lambda x$ with $\lambda \neq 0$, hence $Ax = x/\lambda$ and therefore $A$ maps $H^\pm$ into itself, which is obviously monotone. We write $A^\pm$ for the restriction of $A$ into the subspaces $H^\pm$, then Hille-Yosida theorem gives the existence and uniqueness of solutions to differential equations

$$\begin{cases} \frac{\partial Y}{\partial s} + A^\pm Y = 0, \\ Y(0) = Y_0 \end{cases}$$

on $[0, +\infty)$. What remains is to glue the solutions together to obtain a unique solution defined on $\mathbb{R}$ to the differential equation $\dot{Y}(s) + A_\infty Y = 0$. Set $K(s)$ to be the "kernel" of this differential solution, constructed as

$$K(s) = \begin{cases} e^{-A^+ s} p_+, & s \geq 0; \\ -e^{A^- s} p_-, & s < 0. \end{cases}$$

where $p_\pm$ denotes the orthogonal projection of $H$ onto $H^\pm$, then for $s \neq 0$, $K(s)$ is continuous. It is not continuous at $s = 0$. Define $Q\colon H \to H$ to be

$$Q(Z)(s) = \int_{\mathbb{R}} K(s - \sigma) Z(\sigma) \mathrm{d}\sigma,$$

then $Q$ is linear and if we write $\lambda^\pm$ to be the positive and negative eigenvalues of $A^\pm$ such that it is closest to 0, then we have the estimate

$$\|e^{-A^+ s} p_+ Z(s)\| \leq e^{-\lambda^+ s} \|Z(s)\|$$

and similarly for $e^{-A^- s} p_-$, hence there is a positive integer $\delta > 0$ such that

$$\|K(s)\|_{B(H)} \leq e^{-\delta |s|}$$

here $B(H)$ denotes the set of bounded linear operators on $H$. $Q$ will send $Z$ into $L^2(\mathbb{R}, L^2([0,1], \mathbb{R}^{2n}))$ since

$$\begin{aligned} \|Q(Z)(s,t)\|_{L^2(\mathbb{S})} &\leq \int_{\mathbb{S}} \int_{\mathbb{R}} |K(s-\sigma) Z(\sigma, t)|^2 \mathrm{d}\sigma \mathrm{d}s \mathrm{d}t \\ &\leq \int_{\mathbb{R}} e^{-2\delta|\sigma|} \left( \int_{\mathbb{S}} |Z(\sigma + s, t)|^2 \mathrm{d}s \mathrm{d}t \right) \mathrm{d}\sigma \\ &\leq C \|Z\|_{L^2(\mathbb{S})}. \end{aligned}$$

Now we must show that $Q$ gives the inverse of the operator $L$. Since for each $Y \in L^2(\mathbb{S}, \mathbb{R}^{2n})$ we have

$$Q(Y) = \int_{\mathbb{R}} K(s-\sigma) Y(\sigma, t) \mathrm{d}\sigma = \int_{-\infty}^{s} e^{-A^+(s-\sigma)} Y^+(\sigma, t) \mathrm{d}\sigma - \int_{s}^{+\infty} e^{A^-(s-\sigma)} Y^-(\sigma, t) \mathrm{d}\sigma$$

hence we know that $Q(Y) = Z$ can be decomposed in this way into mutually orthogonal parts $Z = Z^+ + Z^-$, and by taking derivatives, we have

$$\frac{\mathrm{d}Z^+}{\mathrm{d}t} = Y^+(s,t) - \int_{-\infty}^{s} e^{-A^+(s-\sigma)} A^+ Y^+(\sigma, t) \mathrm{d}\sigma = Y^+(s,t) - A^+ Y^+(s,t)$$

and

$$\frac{\mathrm{d}Z^-}{\mathrm{d}t} = Y^-(s,t) - \int_{s}^{+\infty} e^{A^-(s-\sigma)} A^- Y^-(s,t) \mathrm{d}\sigma = -A^- Y^-(s,t) + Y^-(s,t).$$

Combining with these two, we obtain that

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = Y(s,t) - AY(s,t).$$

Therefore we have $L \circ Q = \mathrm{id}$. Conversely, to show $Q \circ L = \mathrm{id}$, for any $Y \in W^{1,2}(\mathbb{S}, \mathbb{R}^{2n})$ we decompose $Y$ as $Y = Y^+ + Y^-$, and by a direct calculation,

$$\begin{aligned}
Q \circ (AY) &= \int_{\mathbb{R}} K(s-\sigma)(AY)(\sigma)\mathrm{d}\sigma = \int_{-\infty}^{s} e^{-A^+(s-\sigma)} A^+ Y^+ \mathrm{d}\sigma - \int_{s}^{+\infty} e^{A^-(s-\sigma)} A^- Y^-(\sigma)\mathrm{d}\sigma \\
&= -\frac{\mathrm{d}}{\mathrm{d}s} \int_{-\infty}^{s} e^{-A^+(s-\sigma)} Y^+(\sigma)\mathrm{d}\sigma + A^+(s) - \frac{\mathrm{d}}{\mathrm{d}s} \int_{s}^{+\infty} e^{A^-(s-\sigma)} Y^-(\sigma)\mathrm{d}\sigma + A^-(s) \\
&= A(s) - \frac{\mathrm{d}}{\mathrm{d}s} \int_{-\infty}^{0} e^{-A^+\sigma} Y^+(\sigma+s)\mathrm{d}\sigma + \frac{\mathrm{d}}{\mathrm{d}s} \int_{0}^{+\infty} e^{A^-\sigma} Y^-(\sigma+s)\mathrm{d}\sigma \\
&= A(s) - \int_{\mathbb{R}} K(-\sigma) \frac{\mathrm{d}Y}{\mathrm{d}s}(s+\sigma)\mathrm{d}\sigma = A(s) + Q\left(\frac{\mathrm{d}Y}{\mathrm{d}s}\right).
\end{aligned}$$

Therefore we have $L \circ Q = \mathrm{id}$. Therefore $L$ is a bijective operator from $W^{1,2}$ to $L^2$, proving the Fredholm property in the case $p = 2$. $\square$

**The General Case $p > 1$.** We want to prove the general case from the special case $p = 2$. To do this, firstly assume $p > 2$, then we have the following improved estimate:

**3.14 Lemma.** *Let $p > 2$. Then there exists a constant $C_1 > 0$ such that for every $k \in \mathbb{R}$ and $Y \in W^{1,p}(\mathbb{S}, \mathbb{R}^{2n})$, we have*

$$\|Y\|_{W^{1,p}([k,k+1]\times[0,1])} \le C_1(\|LY\|_{L^p([k-1,k+2]\times[0,1])} + \|Y\|_{L^2([k-1,k+2]\times[0,1])}.$$

*Proof.* By Hölder's inequality, we readily have $L^p([k-1,k+2]\times[0,1]) \subset L^2([k-1,k+2]\times[0,1])$ and from Sobolev embedding theorem, for all $p > 1$ we have $W^{1,2}([k-1,k+2]\times[0,1]) \subset L^p([k-1,k+2]\times[0,1])$, therefore by using theorem A.10 we have

$$\begin{aligned}
\|Y\|_{W^{1,p}([k,k+1]\times[0,1])} &\le C_1(\|LY\|_{L^p} + \|Y\|_{L^p}) \le C_2(\|LY\|_{L^p} + \|Y\|_{W^{1,2}}) \\
&\le C_3(\|LY\|_{L^p} + \|Y\|_{L^2} + \|LY\|_{L^2}) \le C_4(\|LY\|_{L^p} + \|Y\|_{L^2}). \square
\end{aligned}$$

We could further imporve this result that

**3.15 Lemma.** *There exists a positive constant $C > 0$ such that if $Y \in W^{1,2}(\mathbb{S}, \mathbb{R}^{2n})$ and $LY \in L^p(\mathbb{S}, \mathbb{R}^{2n})$, then $Y \in W^{1,p}(\mathbb{S}, \mathbb{R}^{2n})$ and we have the following estimate*

$$\|Y\|_{W^{1,p}} \le C\|LY\|_{L^p}.$$

*Proof.* If $p > 2$, then Theorem A.10 gives the fact that $Y \in W^{1,p}_{loc}$, and the previous lemma readily gives $Y \in W^{1,p}$. In order to see this estimate, we view $Y$ as a function from $\mathbb{R}$ to $H$, and define the $L^p(\mathbb{R}, H)$-norm to be

$$\|Y\|_{L^p(\mathbb{R},H)} = \left(\int_{\mathbb{R}} \|Y\|_H^p \mathrm{d}s\right)^{\frac{1}{p}}.$$

Now for this $Y$, we have

$$\begin{aligned}
\|Y\|^p_{W^{1,p}([k,k+1]\times[0,1])} &\le C^p(\|LY\|_{L^p([k,k+1]\times[0,1])} + \|Y\|_{L^2([k-1,k+2]\times[0,1])})^p \\
&\le 2^p C^p(\|LY\|^p_{L^p([k-1,k+2]\times[0,1])} + \|Y\|^p_{L^2([k-1,k+2]\times[0,1])}),
\end{aligned}$$

and apply Hölder's inequality,

$$\|Y\|^p_{L^p([k-1,k+2]\times[0,1])} \leq 3^{\frac{p^2}{2(p-2)}} \|Y\|^p_{L^p([k-1,k+2]\times[0,1])}.$$

Summing them up with respect to $k$, we have

$$\|Y\|^p_{W^{1,p}(\mathbb{S})} \leq C(\|LY\|^p_{L^p(\mathbb{S})} + \|Y\|^p_{L^p(\mathbb{R},H)})$$

the only difference between this estimate and the required result is the term $\|Y\|_{L^p(\mathbb{R},H)}$, so we try to give an estimate for this term. Let $Z = LY$, then we have

$$\|Q(Z)\|_{L^p(\mathbb{R},H)} \leq C\|Z\|_{L^p(\mathbb{R},H)}$$

using Young inequality. By Hölder's inequality, we have

$$\|Z\|_{L^p(\mathbb{R},H)} \leq \|Z\|_{L^p(\mathbb{S},\mathbb{R}^{2n})}$$

and hence we obtain the estimate

$$\|Y\|_{L^p(\mathbb{R},H)} \leq C\|LY\|_{L^p(\mathbb{S})}. \qquad \square$$

With this result, it follows directly that $L$ is injective with a closed image in $L^p(\mathbb{S},\mathbb{R}^{2n})$. In order to show bijectiveness, it suffices to show that the image of $L$ is dense in $L^p$, but this follows from the fact that the image of $L$ in the subset $W^{1,2} \cap W^{1,p}$ is already dense in $L^p$.(This is just the subspace $L^2$) Therefore $L$ is bijective. $\square$

**The Case $p > 1$.** The final step is to prove the bijectiveness for $2 > p > 1$. Note that in this case, $L^p$ is the dual space of $L^q$ with $q > 2$ conjugate to $p$, so we consider the dual operator $L^*$ of $L$ defined as $L^* = -\nabla_s + J_0\nabla_t + S(t)$ from $W^{1,q}(\mathbb{S},\mathbb{R}^{2n})$ to $L^q(\mathbb{S},\mathbb{R}^{2n})$. Now assume that $Y \in C_0^\infty(\mathbb{S},\mathbb{R}^{2n})$ satisfying the boundary conditions and by Riesz theorem,

$$\|Y\|_{L^p} = \sup_{\substack{X\in L^p\\ \|X\|=1}} |\langle X, Y\rangle| = \sup_{\substack{X\in L^p\\ \|L^*X\|=1}} |\langle X, LY\rangle| \leq \|LY\|_{L^p} \sup_{\substack{X\in L^p\\ \|L^*X\|=1}} \|X\|_{L^q}.$$

Since the dual operator $L^*$ is bijective, it follows that $\|X\|_{L^q} \leq C\|L^*X\|_{L^q}$ and therefore we have $\|Y\|_{L^p} \leq C\|LY\|_{L^q}$, and for partial derivatives of $Y$, we have a similar estimate combining to an estimate for $\|Y\|_{W^{1,q}}$:

$$\|Y\|_{W^{1,q}(\mathbb{S},\mathbb{R}^{2n})} \leq C\|LY\|_{L^q},$$

and therefore $L$ is injective with closed image. To show surjectivity here, consider the cokernel of $L$, which is the kernel of $L^*$, and since $L^*$ is bijective, it follows that the cokernel of $L$ is trivial, hence $L$ is bijective. $\square$

Therefore we have proved theorem 3.10 for all $p > 1$. $\square$

**3c) Transversality** In this paragraph we prove the vital theorem for determining the dimension of moduli spaces: the transversality property. The main conclusion is that

**3.16 Theorem** (Transversality)**.** For a generic choice of almost complex structures, the Cauchy-Riemann operator $\bar{\partial}$ defined as a section of the Banach bundle $\mathcal{L}^p(x,y) \to \mathcal{P}^{1,p}(x,y)$ has a surjective linearization and the linearization admits a right inverse at the zero set. Therefore $\mathcal{M}_J(x,y)$, for a generic choice of $J$, is finite-dimensional and its dimension equals the Fredholm index of $\bar{\partial}$.

In order to prove this, we need the "generic" choice of an almost complex structure, which is non-constructible, hence for each given almost complex structure $J$, we introduce the space of "perturbations" of $J$ so that it serves as the tangent space to the space of all almost complex structures of at this point $J$. Then we could view the Cauchy-Riemann operator $\bar{\partial}_*$ as a section of the product manifold of $\mathcal{P}^{1,p}(x,y)$ and $C^\infty(P;[0,1]\times\mathcal{J})$(where the subscript $*$ denotes the choice of almost complex structure). Then recall from the previous sections that for any choice of almost complex structure $J$, we have the operator $\bar{\partial}$ is Fredholm, and from the following abstract result in functional analysis:

**3.17 Proposition.** Assume that $X, Y, Z$ are Banach spaces, $F\colon X\oplus Y \to Z$ is a surjective bounded linear operator such that $F = f \oplus g$ where $f\colon X \to Z$ is Fredholm, then $F$ admits a right inverse.

*Proof.* Since $f$ is Fredholm, $\mathrm{Im} f$ is closed and $\operatorname{coker} f$ is finite-dimensional, hence there exists a finite-dimensional subspace $Z_1 \subseteq Z$ such that $Z = Z_1 \oplus \mathrm{Im} f$. Since $F$ is surjective, $\mathrm{Im} g \supseteq Z_1$ and since $\ker f$ is also finite-dimensional, there is a complementary subspace $X_1$ of $\ker f$ in $X$ and hence we can further decompose into $\ker f \oplus X_1 \oplus Y$. Then $f$ is an isomorphism from $X_1$ onto $\mathrm{Im} f$, and for elements in $Z_1$, we assume $Z_1 = \mathrm{span}\{z_1, z_2, \cdots, z_k\}$. Define a map $G\colon Z \to X \oplus Y$ by setting $G(z_i) = y_i$ for some $y_i \in Y$ such that $g(y_i) = z_i$ and $G(x) = f^{-1}(x)$ for $x \in \mathrm{Im} f$. Since $f^{-1}$ is continuous and $Z_1$ is finite-dimensional, $G$ is continuous and $F \circ G$ is obviously the identity map, thus $G$ is the right inverse to $F$. □

we know that it suffices to show that the operator $\bar{\partial}_*$ is surjective. Floer tried to give a proof in his paper [Flo88d], but it has some errors, and was proved by Oh in the appendix in his paper [Oh06]. Here we present the proof in Oh's paper.

*Proof of Theorem 3.16.* The tangent space to $C^\infty(P,[0,1]\times\mathcal{J})$ at some point $J$ is given explicitly by

$$T_J C^\infty([0,1]\times\mathcal{J}(P,\theta)) = \{j_t\colon \mathbb{S} \to \mathrm{End}(TP) | g(j_t v, w) + g(v, j_t w) = 0, j_t J + J j_t = 0, 0 \le t \le 1\}.$$

We introduce a norm $\|\cdot\|_\varepsilon$ on some proper subspace of $T_J C^\infty([0,1]\times\mathcal{J}(P,\theta))$ for an infinite family of positive numbers $\varepsilon = (\varepsilon_k)_{k\in\mathbb{Z}_{\ge 0}}$. For simplicity, we write this tangent space as $C^\infty(P,[0,1]\times\mathcal{J}_J(P))$. The norm is given by

$$\|j\|_\varepsilon = \sum_{k\ge 0} \varepsilon_k \max_{x\in P} |\mathrm{d}^k j(x)|,$$

and we set

$$C^\varepsilon([0,1]\times\mathcal{J}_J) = \{j | \|j\|_\varepsilon < \infty\}$$

as the subspace of $C^\infty$.

**3.18 Lemma.** *The space $C^\varepsilon$ is a dense subspace of $L^2$.*

For fixed positive number $r > 0$ and a prescribed open neighbourhood $U$ of $L_0 \cap L_1$, we set $C^\varepsilon_r([0,1] \times \mathcal{J}_J)$ to be the set of all $j$s such that $\|j\|_\varepsilon < r$ and $j(t,x) \equiv 0$ for all $x \in U$. Set $\mathcal{L}$ to be the pull-back bundle of $\mathcal{L}^p$ via the projection map $\mathcal{P}^{1,p} \times C^\varepsilon_r$, where $\mathcal{P}^{1,p}$ is the "universal" space of trajectories, i.e. the set of all smooth maps $u \colon \mathbb{S} \to P$ such that $u(\mathbb{R} \times \{i\}) \subset L_i$, and $\mathcal{L}^p$ is the corresponding "universal" bundle. Then the differential of the section $\bar\partial_*$ on any pair $(u,j)$ decomposes into two parts, written $\mathrm{d}_1\bar\partial$ and $\mathrm{d}_2\bar\partial$:

$$\mathrm{d}_1\bar\partial_{(u,j)} := \mathrm{D}_{(u,j)}; \quad \mathrm{d}_2\bar\partial_{(u,j)} = (\mathrm{d}\exp_J)_j \xi \dot u$$

where we write D for the linearized Cauchy-Riemann operator, and $\exp_J$ the exponential map at $J$. From the Fredholm property 3.10 we know that $\mathrm{D}_{(u,j)}$ is Fredholm, hence it suffices to show that the linearization $\mathrm{d}\bar\partial$ is surjective. Note that the linearization $\mathrm{d}\bar\partial_{(u,j)}$ has closed image since $\mathrm{D}_{(u,j)}$ is Fredholm, hence it suffices to show the image of $\mathrm{d}\bar\partial$ is dense in $L^p(u)$. Assume to the contrary that this is not true, then there exists a non-zero linear functional $\gamma$ on $L^p(u)$ such that $\gamma|_{\mathrm{Im d}\bar\partial_*} = 0$. Then for all $\xi \in T_u\mathcal{P}$ and all $\eta \in T_j\mathcal{J}$ we have

$$\gamma(\mathrm{D}_{(u,j)}\xi) = \gamma((\mathrm{d}\exp_J)_j \eta \dot u) = 0.$$

From the standard theory of real analysis([Rud87]) we have $\gamma \in L^q(u)$ and hence there exists a function in $L^q(u)$, still denoted by $\gamma$, such that

$$\gamma(\xi) = \int_{\mathbb{S}} g(\gamma(z), \xi(z)) \mathrm{d}\mu_g$$

for all $\xi \in L^p(u)$. Now the first equality $\gamma(\mathrm{D}_{(u,j)}\xi) = 0$ implies that $\mathrm{D}^*_{(u,j)}\gamma = 0$ as distributions, hence by elliptic regularity we know that $\gamma$ is at least continuous. To conclude, we need a result that any pesudo-holomorphic strip is "somewhere injective":

**3.19 Lemma.** *The set $R(u)$ of all points $(s,t) \in \mathbb{S}$ such that $\dfrac{\partial u}{\partial s}(s,t) \neq 0$, $u(s,t) \notin L_0 \cap L_1$ and $u(s,t) \notin u((\mathbb{R} \setminus \{s\}) \times \{t\})$ for all $t \in [0,1]$ is dense and open in $\mathbb{S}$.*

Hence if we could show that $\gamma(s,t) \equiv 0$ in $R(u)$, then we could conclude that $\gamma \equiv 0$ on $\mathbb{S}$. Assume to the contrary that there exists some $(s_0, t_0)$ with $\gamma(s_0,t_0) \neq 0$, then by definition of $R(u)$ there are no points $(s, t_0)$ with $s \neq s_0$ but $u(s,t_0) = u(s_0,t_0)$. By injectivity of $u$ it follows that $\dot u_{s_0} \neq 0$ on the whole inteval and hence the preimage of $u(s_0,t_0)$ on $\mathbb{S}$ is finite. Therefore we could choose a small element $\eta \in T_jC^\varepsilon_r$ supported in a disjoint union of closed neighbourhoods of each preimage such that

$$g((\mathrm{d}\exp_{J(s,t)})_{j(s,t)}\eta(s,t)\dot u(s,t), \gamma(s,t)) > 0$$

in the interior of this support, and therefore by taking integration, we obtain a contradiction. Now since $\mathrm{d}\bar\partial$ is surjective and admits a continuous right inverse, we consider the projection map $\pi_J \colon \mathcal{Z}(x,y) \to \mathcal{J}_J(P,\theta)$ where $\mathcal{Z}(x,y)$ is the intersection of $\bar\partial_*$ with the zero-section. From the above transversality argument it follows that $\mathcal{Z}(x,y)$ is a Banach manifold and the linearization of $\pi_J$ at any point $u$ has kernel equal to the kernel of $(\mathrm{d}_1\bar{\partial}_*)_u$, which is finite-dimensional and the image is exactly the inverse image of $\mathrm{d}_1\bar{\partial}_{(u,j)}$ via the map $\mathrm{d}_2\bar{\partial}_{(u,j)}$. Hence the cokernel is also finite, and hence the map $\pi_J$ is a Fredholm map. From Sard-Smale theorem [Sma65], it follows that for a generic choice of $j$, the linearization $\mathrm{d}_1\bar{\partial}_{(u,j)}$ is surjective, i.e. for a generic choice of $J'$(push $j$ via the exponential map), the linearization $(\mathrm{d}\bar{\partial}_{J'})_u$ is surjective on its solution set, hence the preimage would be a finite-dimensional manifold with index equal to the Fredholm index of this linearized operator. ☐

It remains to prove the somewhere injectivity property 3.19. To do this, we add the requirements one by one and show that all these conditions are open and dense conditions on $\mathbb{S}$.

**3.20 Lemma.** *The set*

$$R(u) = \left\{ s \in \mathbb{R} \middle| \frac{\partial u}{\partial s}(s,t) \neq 0 \textit{ for all } t \in [0,1] \right\}$$

*is open and dense in* $\mathbb{R}$.

*Proof.* The condition is an open condition, hence it suffices to show this is dense. Assume to the contrary that there is a bounded closed inteval $I \subseteq \mathbb{R}$ such that for all $s \in I$, there is $t_s \in [0,1]$ such that $\frac{\partial u}{\partial s}(s,t_s) = 0$, then since $I \times [0,1]$ is compact, there is a cumulative point $(s_0,t_0)$ of the set $\{(s,t_s)\}_{s\in I}$. Since $\frac{\partial u}{\partial s}$ satisfies the linearized Cauchy-Riemann equation (19), we can apply the unique continuation theorem A.17 to $\frac{\partial u}{\partial s}$ to conclude that $\frac{\partial u}{\partial s} \equiv 0$ on the whole of $\mathbb{S}$, a contradiction. Therefore $R(u)$ is open and dense. ☐

Next, we impose the condition that the image of $u$ should be disjoint from the intersection points in $L_0 \cap L_1$.

**3.21 Lemma.** *The set*

$$\bar{R}(u) = \{s \in \mathbb{R} | \mathrm{Im} u_s \cap L_0 \cap L_1 = \emptyset\}$$

*is open and dense in* $R(u)$.

*Proof.* Since $L_0 \pitchfork L_1$, the intersection points are discrete and finite, hence $\bar{R}(u)$ must be open and dense in $R(u)$. To prove this is dense in $R(u)$, assume to the contrary that there is a bounded closed interval $I \subset \mathbb{R}$ such that for all $s \in I$, $\mathrm{Im} u_s \cap L_0 \cap L_1 \neq \emptyset$, i.e. there exists $t_s$ for each $s \in I$ such that $u(s,t_s) \in L_0 \cap L_1$, then again there is an element $(s_0,t_0)$ such that $u(s_0,t_0) \in L_0 \cap L_1$ and there is a sequence $(s_n, t_{s_n}) \xrightarrow{n\to\infty} (s_0,t_0)$. Since $L_0 \cap L_1$ is discrete, it follows that for $n$ sufficiently large we must have $u(s_n,t_n) = u(s_0,t_0)$, hence $\frac{\partial u}{\partial s}(s_0,t_0) = 0$, a contradiction to the fact that $s_0 \in R(u)$. ☐

By some abuse of notations, let's write $R(u)$ for $\bar{R}(u)$ in the previous lemma, and step forward to add conditions in the set $R(u)$.

**3.22 Lemma.** *The set*

$$\bar{R}(u) = \{s \in R(u) | u_s(t) \subsetneq u((\mathbb{R} \setminus \{s\}) \times \{t\}) \ \forall t \in [0,1]\}$$

*is open and dense in $R(u)$.*

*Proof.* In this lemma even the openness is not obvious. Assume to the contrary that there is some point $s_0 \in \bar{R}(u)$ such that there is a sequence $\{s_n\} \subset (\mathbb{R} \setminus \bar{R}(u))$ that tends to $s_0$. By definition, for each such $s_n$ and any $t \in [0,1]$, there exists $s'_n \neq s_n$ such that $u(s_n, t) = u(s'_n, t)$. Since $\{s_n\}$ is bounded, $\{(s'_n, t)\}$ admits a convergent subsequence, still written $\{(s'_n, t)\}$, that tends to some point $(s'_0, t)$. By construction, $u(s_0, t) = u(s'_0, t)$ and since this holds for all $t \in [0,1]$, from definition of $\bar{R}(u)$ it follows that $s_0 = s'_0$, hence we conclude that $\dfrac{\partial u}{\partial s}(s_0, t) = 0$, a contradiction. Therefore $\bar{R}(u)$ is open in $R(u)$.

Now we prove that $\bar{R}(u)$ is dense in $R(u)$. Again assume to the contrary that there exists an closed inteval $I \subset R(u)$ such that $\bar{R}(u) \cap I = \emptyset$, then since $u$ is a local embedding, there exists a closed subinteval $I_1 \subset I$ such that $u(I_1 \times \{t\}) \subseteq u((\mathbb{R} \setminus I_1) \times \{t\})$ for all $t \in [0,1]$. In particular, this holds for $t = 0$, hence there is a distinct closed inteval $I_2 \subset R(u)$ such that $u(I_1 \times \{0\}) = u(I_2 \times \{0\})$. We could then pick open neighbourhoods $A_1$ of $I_1 \times \{0\}$ and $A_2$ of $I_2 \times \{0\}$ such that $u(A_1) = u(A_2)$. Now assume that $u$ is injective in $A_1$, then we can construct a biholomorphic map $h \colon A_1 \to A_2$ and if we pick $x \in I_1$ with a path $\gamma_1(t) = (x, t)$, then we can extend $h$ to a neighbourhood $\Gamma_1$ of $\gamma$ in $I \times [0,1]$ containing $A_1$ and its image would be extended to a neighbourhood $\Gamma_2$ of a corresponding path $\gamma_2$. Now we could glue $\mathbb{S}$ along $h$ and obtain a cylinder, contradicting the fact that

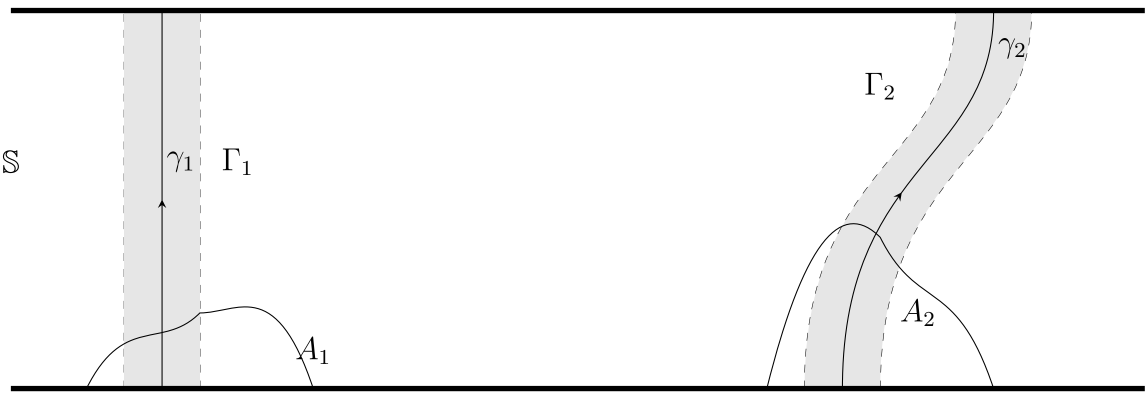


Figure 3: Glueing in the Strip

the region $\Gamma_1$ must be far away from the intersection points in $L_0 \cap L_1$. Therefore we must have $\bar{R}(u)$ dense in $R(u)$. □

We replace again $\bar{R}(u)$ by $R(u)$, and a final argument is that

*Proof of Lemma 3.19.* We write $\bar{R}(u)$ the set we defined in the lemma. Observe that $\bar{R}(u) = R(u) \times [0,1]$, therefore $\bar{R}(u)$ is dense and open in $\mathbb{S}$. □

**3d) Dimension of Moduli Spaces** Here in order to introduce cohomology, we write this notation for the space of trajectories starting from $x$ and tending to $y$. We will show that for each space $\mathcal{M}(y,x)$, the dimension $\dim \mathcal{M}(y,x)$ for a generic choice of $J$, will be the **Maslov-Viterbo index** $\operatorname{Ind}(x,y)$ of any pseudo-holomorphic disk $u \in \mathcal{M}(x,y)$. This is an index depending only on $x$ and $y$, and we could define a function $\mu\colon \operatorname{Crit}(\mathcal{A}) \to \mathbb{Z}$ such that $\mu(x)-\mu(y) = \operatorname{Ind}(x,y)$, and this is defined up to an additive constant. Thus if we assume that $\mu(x) - \mu(y) = 1$, then the space $\mathcal{M}(y,x)$ is one-dimensional. Note that this is the space of trajectories from $x$ to $y$, and we have a natural group action $\mathbb{R} \curvearrowright \mathcal{M}(x,y)$ by translation on the $s$-variable, thus to obtain the space of "geometrically distinct" trajectories, we must quotient the space by $\mathbb{R}$ and obtain a zero-dimensional moduli space $\widehat{\mathcal{M}}(y,x) = \mathcal{M}(y,x)/\mathbb{R}$. Assuming a compactness property, it follows that $|\widehat{\mathcal{M}}(y,x)| < \infty$ and hence we can count the number of trajectories from $x$ to $y$. Now we define the **Floer differential** $\partial_F$ to be the map

$$\partial_F x = \sum_{y\in\operatorname{Crit}(\mathcal{A}),\mu(y)=\mu(x)+1} \#\mathcal{M}(x,y)y$$

where the coefficient is given modulo 2.

**The Generalized Riemann-Roch Theorem** The calculation of dimension was done originally via the analysis of spectral flow in Floer's paper [Flo88a], inspired by Atiyah, Patodi and Singer's famous paper [APS75]. But in this paper we want to give an alternative approach to this calculation, which relies on a generlized Riemann-Roch theorem discovered by Gromov, which tells the relationship between Fredholm operators of the real linear Cauchy-Riemann operator on a complex vector bundle over some Riemann surface $\Sigma$ possibly with boundary and the euler characteristic of $\Sigma$ as well as the relative(absolute) Chern number of a pair $(E,F)$ where $F$ is a totally real subbundle of $E|_{\partial\Sigma}$.

Before stating the main theorem, let's review some basis of complex analysis. Given an (almost) complex structure $j$ on $\Sigma$, one can associate an **hermitian inner product** which is a sesquilinear form

$$h\colon T_{\mathbb{C}}\Sigma \otimes T_{\mathbb{C}}\Sigma \to C^\infty(\Sigma;\mathbb{C})$$

such that $h(j\cdot, j\cdot) = h(\cdot,\cdot)$. Here $T_{\mathbb{C}}\Sigma$ is the complexification of the real tangent bundle $T\Sigma$, and we say $h$ is **sesquilinear** if it is complex-linear with respect to the first variable and complex-antilinear with respect to the second. We could then decompose $h$ into its real and imaginary parts(see [Wel08]), with its real parts a real inner product such that $J$ is orthogonal, and imaginary parts a 2-form. We write $\langle\cdot,\cdot\rangle$ for the real parts of $h$ and we call it an **hermitian structure** over $\Sigma$.(It's not hard to recover $h$ from its hermitian structure. See again [Wel08].) There is a natural generalization of the hermitian structure to the case of a smooth complex vector bundle $E \to \Sigma$. On this vector bundle, we write the bundle of differential forms of type $(p,q)$ with values in $E$ by $\Omega^{p,q}(\Sigma, E) := E \otimes \Omega^{p,q}(\Sigma)$, and similarly for differential $k$-forms with values in $E$ by $\Omega^k(\Sigma, E)$.

**3.23 Definition.** Given a connection $\nabla\colon \Omega^0(\Sigma,E) \to \Omega^1(\Sigma,E)$, we say it is an **hermitian connection** if for any open subset $U \subset \Sigma$ and any vector field $\xi, \eta \in$

$\Gamma(U, E)$ and functions $f \in C^\infty(U, \mathbb{C})$ we have

$$\nabla(f\xi) = f\nabla\xi + \mathrm{d}f \otimes \xi \quad \text{and } \mathrm{d}\langle \xi, \eta\rangle = \langle \nabla\xi, \eta\rangle + \langle \xi, \nabla\eta\rangle,$$

where on the right-hand side the inner product $\langle \cdot, \cdot \rangle$ is the bilinear map $E \otimes T_{\mathbb{C}}\Sigma \otimes E \to T_{\mathbb{C}}\Sigma$ by contraction.

From [Wel08] we konw that such an hermitian connection is unique for the given vector bundle $E$. For this hermitian operator, we can consider its anti-lienar part $\nabla^{0,1}$, which is a differential operator on $E$ with symbol $\bar{\partial}$, thus it follows that such an hermitian structure determines a **Cauchy-Riemann operator** on $E$, and we can further show that any Cauchy-Riemann operator comes in this way. Here we just define the Cauchy-Riemann operator $D$ to be the complex linear first-order differential operator on $E$ with symbol $\bar{\partial}$. We say $D$ is a **real linear Cauchy-Riemann operator** if it is of the form $D_1 + \alpha$ with $D_1$ a Cauchy-Riemann operator and $\alpha$ a real 1-form.

**The Boundary Maslov Index** The Fredholm index of such a Cauchy-Riemann operator corresponds to the so-called **boundary Maslov index** on $\Sigma$, which is an index for the bundle pair $(E, F)$. It's a slight modification of the classical Maslov index, which was defined by Maslov who wanted to compute the coefficient of the asymptotic expansion in semiclassical approximation, which gives the quantization conditions. It was then developed by Arnold in his paper [Arn67] who showed that the Maslov index is obtained by counting the intersection number of a loop of Lagrangian subspaces with the degenerate part of a **Lagrangian Grassmannian** $L(n)$ that intersects with a given Lagrangian subspace non-trivially. Viterbo then use this index in his paper [Vit87] to define the well-known Maslov-Viterbo index, for a given transversal pair of Lagrangian submanifolds $(L_0, L_1)$ and two intersection points $x, y$. Robbin and Salamon wrote a systematic treatment with Maslov index in their paper We start with a review of the construction of Maslov index. Let $R(n) := \mathrm{GL}(n, \mathbb{C})/\mathrm{GL}(n, \mathbb{R})$ be the homogeneous space of all totally real submanifolds over the standard Euclidean space $\mathbb{C}^n$, and let $L(n) = \mathrm{Sp}(2n)/\mathrm{GL}(n, \mathbb{R})$ be the Lagrangian Grassmannian, i.e. the space of all Lagrangian submanifolds inside the standard symplectic space $\mathbb{R}^{2n}$, here $\mathrm{Sp}(2n)$ is the group of linear symplectomorphisms over $\mathbb{R}^{2n}$.

**3.24 Proposition.** $L(n)$ is the deformation restract of $R(n)$ and the determinant map

$$\det{}^2\colon L(n) \to \mathbb{S}^1$$

gives an isomorphism $\det^2_*\colon \pi_1(L(n)) \to \pi_1(\mathbb{S}^1)$, thus we know that $\pi_1(L(n)) \cong \mathbb{Z}$.

*Proof.* We use the polar decomposition in [Hal15] again for $\mathrm{Sp}(2n, \mathbb{R})$ and for $\mathrm{GL}(k, \mathbb{R})$ to deduce that $\mathrm{Sp}(2n, \mathbb{R})$ deformation retracts to $U(n)$ and $\mathrm{GL}(k, \mathbb{R})$ deformation restracts onto $\mathrm{O}(n)$, and we have that $L(n) = \mathrm{U}(n)/\mathrm{O}(n)$. Note that $\mathrm{GL}(n, \mathbb{C})$ also deformation retracts onto $\mathrm{U}(n)$, it follows that $L(n)$ is a deformation retract of $R(n)$. Now consider the determinant map

$$\det\colon \mathrm{U}(n) \to \mathbb{S}^1$$

which is a group homomorphism with kernel $\mathrm{SU}(n)$, thus we have a natural fibration $\mathrm{SU}(n) \to \mathrm{U}(n) \to \mathbb{S}^1$ which induces an exact sequence on homotopy groups

$$0 = \pi_1(\mathrm{SU}(n)) \to \pi_1(U(n)) \xrightarrow{\det_*^2} \pi_1(\mathbb{S}^1) \to \pi_0(\mathrm{SU}(n)) = 0 \tag{7}$$

It follows directly that $\det_* \colon \pi_1(U(n)) \to \pi_1(\mathbb{S}^1)$ induces an isomorphism of groups, hence $\pi_1(U(n)) \cong \mathbb{Z}$. Since $L(n) = \mathrm{U}(n)/\mathrm{O}(n)$, it follows that only when square the map det can we pass to the quotient and obtain a well-defined continuous map $\det^2 \colon L(n) \to \mathbb{S}^1$. To show this is an isomorphism in fundamental groups, note that $\det^2$ gives a long exact sequence for the map $\mathrm{U}(n) \to \mathbb{S}^1$ as

$$0 = \pi_1(\widetilde{\mathrm{SU}(n)}) \to \pi_1(U(n)) \xrightarrow{\det_*^2} \pi_1(\mathbb{S}^1) \to \pi_0(\widetilde{\mathrm{SU}(n)}) \to \pi_0(\mathrm{U}(n)) = 0 \tag{8}$$

where $\widetilde{\mathrm{SU}(n)}$ is the trivial two-fold covering of $\mathrm{SU}(n)$, while as a group it is a semi-direct product of $\mathrm{SU}(n)$ with $\mathbb{Z}/2\mathbb{Z}$, hence its 0-dimensional homotopy group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. As a homogeneous space, $L(n)$ admits a natural long exact sequence of homotopy groups

$$\pi_1(\mathrm{O}(n)) \to \pi_1(\mathrm{U}(n)) \to \pi_1(L(n)) \to \pi_0(\mathrm{O}(n)) \to \pi_0(U(n)) = 0 \tag{9}$$

since $\pi_1(\mathrm{O}(n)) = \pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}/2\mathbb{Z}$ when $n \geq 3$, trivial when $n = 1$ and is isomorphic to $\mathbb{Z}$ when $n = 2$, it follows that when $n \geq 3$ we have $\pi_1(\mathrm{SO}(n)) \to \pi_1(\mathrm{U}(n))$ trivial, thus when $n \geq 3$, this gives a short exact sequence

$$0 \to \pi_1(\mathrm{U}(n)) \to \pi_1(L(n)) \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Since $\widetilde{\mathrm{SU}(n)}$ is a two-fold covering of $\mathrm{O}(n)$, we have a natural group homomorphism $\pi_0(\widetilde{\mathrm{SU}(n)}) \to \pi_0(\mathrm{O}(n))$ which combining with (9) and (8) gives the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(\mathrm{U}(n)) \arrow[r, "\det_*^2"] \arrow[d, equal] & \pi_1(\mathbb{S}^1) \arrow[r] & \pi_0(\widetilde{\mathrm{SU}(n)}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_1(\mathrm{U}(n)) \arrow[r] & \pi_1(L(n)) \arrow[r] \arrow[u, "\det_*^2"] & \pi_0(\mathrm{O}(n)) \arrow[r] & 0
\end{tikzcd}.$$

By definition we know that the first square commutes, and by the construction of the induced connecting homomorphism from $\pi_1(X, A)$ to $\pi_0(A)$ in Hatcher's book [Hat02], it follows that the second square is also commutative, with the last column an isomorphism, thus we can reverse the arrow and obtain from five-lemma that $\det_*^2$ in the middle column is an isomorphism. $\square$

**3.25 Definition.** For a loop $\Lambda \colon \mathbb{S}^1 \to L(n)$, we define its **Maslov index**, denoted $\mu(\Lambda)$, to be the degree of the map $\det^2 \circ \Lambda \colon \mathbb{S}^1 \to \mathbb{S}^1$.

From the above Proposition it follows directly that

**3.26 Corollary.** Two loops $\Lambda_1, \Lambda_2 \colon \mathbb{S}^1 \to L(n)$ are homotopic if and only if they have the same Maslov index.

Since $R(n)$ deformation retracts onto $L(n)$, by a suitable refinement of determinant, it follows that the Maslov index $\mu(\Lambda)$ for a loop $\Lambda \colon \mathbb{S}^1 \to R(n)$ is well-defined.

**3.27 Corollary.** Two loops $\Lambda_1, \Lambda_2 \colon \mathbb{S}^1 \to R(n)$ are homotopic if and only if they have the same Maslov index.

Now we consider a compact Riemann surface $\Sigma$ possibly with boundary $\partial\Sigma$, with a given bundle pair $(E, F)$ over $\Sigma$. Since $\Sigma$ is compact, the boundary $\partial\Sigma$ is a bunch of disjoint circles, thus giving a map $\Gamma \colon \partial\Sigma \to F$. Note that in this case, with the almost complex structure $J$ on $E$, the space $(E_x, J_x)$ varies over $\Sigma$, and hence we cannot apply the above construction directly in this case. However, since the boundary $\partial\Sigma$ consists only of circles, we can pick a trivialization $\Phi \colon \partial\Sigma \times \mathbb{C}^n \to E|_{\partial\Sigma}$ of $E$ on $\partial\Sigma$ and in the trivialization, the totally real subbundle $F$ is given by a smooth map $\Lambda \colon L(n)$ with $z \mapsto \Phi_z^{-1}(F_z)$.

**3.28 Definition.** The boundary Maslov index of the bundle pair $(E, F)$ over a Riemann surface $\Sigma$ is given by $\mu(E, F) := \mu(\Lambda)$ with $\Lambda$ defined as above. When $\partial\Sigma$ is empty, we define $\mu(E, F) := 2c_1(E)$ where $c_1$ is the first Chern class of $E$.

We can view the boundary Maslov class as the characteristic class(obstruction to the pair $(E, F)$ being trivial) of the bundle pair $(E, F)$, so we also say this is a **relative Chern class**.(For example, in Seidel's paper [Sei00]) However, we need some effort to show that this defintion is well-defined, i.e. it is independent of the choice of trivializations on the boundary, and to deduce some properties needed for calculation. We say two bundle pairs $(E_1, F_1)$ and $(E_2, F_2)$ over the same Riemann surface are **isomorphic** if there is an isomorphism sending $F_1$ to $F_2$. Since $F_i$ are totally real, we have $E_i \cong F_i \oplus JF_i$ and hence the isomorphism on $F_i$ extends to an isomorphism on $E_i$.

**3.29 Theorem.** There exists a unique integer-valued function $\mu$ on the class of all bundle pairs $(E, F)$ such that the following properties hold:

Isomorphism If $\Phi \colon E_1 \to E_2$ is a vector bundle isomorphism covering an orientation-preserving diffeomorphism $\phi \colon \Sigma_1 \to \Sigma_2$, then $\mu(E_1, F_1) = \mu(E_2, \Phi(F_1))$.

Direct Sum $\mu(E_1 \oplus E_2, F_1 \oplus F_2) = \mu(E_1, F_1) + \mu(E_2, F_2)$.

Composition We say a decomposition of a bundle pair $(E_{02}, F_{02})$ over a Riemann surface $\Sigma_{02}$ is a decomposition of $\Sigma_{02}$ into $\Sigma_{01}$ and $\Sigma_{12}$(See Definition B.1) with the induced subbundles $E_{01} = E_{02}|_{\Sigma_{01}}$, $E_{12} = E_{02}|_{\Sigma_{12}}$ and a totally real subbundle $F_1$ on $\Sigma_{01} \cap \Sigma_{12}$ such that $F_{01} = F_0 := F_{02}|_{\partial\Sigma_{02} \cap \Sigma_{01}} \cup F_1$ and $F_{02} = F_2 := F_{02}|_{\partial\sigma_{02} \cap \Sigma_{12}} \cup F_1$. For such a decomposition, we have

$$\mu(E_{02}, F_{02}) = \mu(E_{01}, F_{01}) + \mu(E_{12}, F_{12}).$$

Normalization For $\Sigma = \mathbb{D}$ the closed unit disk, $E = \mathbb{D} \times \mathbb{C}$ the trivial bundle, and $F_z = \mathbb{R}e^{ik\theta/2}$ for $z = e^{i\theta} \in \partial\mathbb{D} = \mathbb{S}^1$ we have $\mu(\mathbb{D} \times \mathbb{C}, F) = k$.

**3.30 Theorem.** The boundary Maslov index $\mu$ satisfies the following further properties:

Trivial Bundle If $\partial\Sigma \neq \emptyset$ and $E = \Sigma \times \mathbb{C}^n$, then $\mu(\Sigma \times \mathbb{C}^n, F) = \mu(\Lambda)$ where $\mu$ is the Maslov index of the loop $\Lambda(z) = F_z$ for $z \in \partial\Sigma$.

Chern Class If $\partial\Sigma = \emptyset$, then $\mu(E, \emptyset)$ is twice the value of the first Chern class $c_1(E) \in H^2(\Sigma)$ on the fundamental class $[\Sigma] \in H_2(\Sigma)$: $\mu(E, \emptyset) = 2\langle c_1(E), [\Sigma] \rangle$.

Moreover, we can deduce that

**3.31 Corollary.** Tow bundle pairs $(E_1, F_1)$ and $(E_2, F_2)$ over the same Riemann surface $\Sigma$ are isomorphic if and only if $E_1$ and $E_2$ have the same rank, $\mu(E_1, F_1) = \mu(E_2, F_2)$, and for each connected component $C$ of the boundary $\partial\Sigma$ the real vector bundles $F_1|_C$ and $F_2|_C$ are isomorphic.

In order to give a proof, note that in dimension 2, complex vector bundles over a Riemann surface with boundary can always be trivialized since the homotopy type of a Riemann surface with boundary is a bouquet of circles. But then we need to decompose the totally real subbundle $F$ so that we could apply the direct sum axiom to give boundary Maslov indices for all bundle pairs. This leads us to the following normalization of bundle pairs:

**3.32 Proposition** (Normalization of Bundle Pairs)**.** For any bundle pair $(E, F)$ over $\mathbb{D}$, we say a **partial framing** of $F$ is a pair $(L; s_1, \cdots, s_{n-1})$ such that $s_1, \cdots, s_{n-1}$ are nowhere vanishing sections of $F$ and $L$ a line subbundle of $F$ such that $F = L \oplus \bigoplus_{i=1}^{n-1} \mathbb{R}s_i$. Equipped with such a partial framing, there exists a trivialization

$$\Phi\colon E \xrightarrow{\simeq} \Sigma \times \mathbb{C}^n$$

and a smooth function $\lambda\colon \partial\Sigma \to \mathbb{S}^1$ such that $\Phi_z(L_z) = \sqrt{\lambda(z)}\mathbb{R}$ for all $z \in \partial\Sigma$ and $\Phi(s_i) = e_i$ where $\{e_i\}$ is the standard basis of $\mathbb{C}^n$. In particular, for all $z \in \partial\Sigma$, we have $\Phi(F_z) = \sqrt{\lambda(z)}\mathbb{R} \oplus \mathbb{R}^{n-1}$.

*Proof.* For sufficiently small positive number $\varepsilon$ let $\bar{U}_\varepsilon$ be the closure of a collar neighbourhood of $\partial\Sigma$, then it follows that $\Sigma \setminus U_\varepsilon = \Sigma_1$ is diffeomorphic to $\Sigma$, and for the bundle $E|_{\Sigma_1}$, we have a natural trivialization $E|_{\Sigma_1} \xrightarrow{\Psi_1} \Sigma_1 \times \mathbb{C}^n$. Since $\bar{U}_\varepsilon$ is diffeomorphic to a cylinder with ends $\partial\Sigma$ and $\partial\Sigma_1$, $E \to \bar{U}_\varepsilon$ admits a trivialization $E|_{\bar{U}_\varepsilon} \xrightarrow{\Psi_2} \bar{U}_\varepsilon \times \mathbb{C}^n$. We then have a loop $\Lambda_2\colon \partial\Sigma \to R(n)$ given by $\Lambda_0(z) = \Psi_2(F_z)$. Pick $\lambda(z)$ a smooth function and write $\Lambda_1\colon \partial\Sigma_1 \to R(n)$ be the totally real subbundle of $E|_{\partial\Sigma_1}$ given by $\Lambda_1(z) = \sqrt{\lambda(z)}\mathbb{R} \oplus \mathbb{R}^{n-1}$ such that $\mu(\Lambda_1) = \mu(\Lambda_0)$. Then Corollary 3.26 gives a homotopy

$$\Gamma\colon [0,1] \times \partial\Sigma \to R(n)$$

such that $\Gamma(0, -) = \Lambda_0(-)$ and $\Gamma(1, -) = \Lambda_1(-)$. This in fact gives a smooth family of operators $U\colon [0,1] \times \partial\Sigma \to \mathrm{GL}(n, \mathbb{C})$ such that $U(1, z)$ is contractible and $\Lambda_0(z) = U(1, z)\Lambda_1(z)$, $U(0, z) \equiv \mathrm{id}$. Therefore we set $\Psi_U\colon \bar{U}_\varepsilon \times \mathbb{C}^n \to \bar{U}_\varepsilon \times \mathbb{C}^n$ given by $\Psi_U(x, v) = (x, U(x)v)$, then the composition $\Psi_U \circ \Psi_2$ gives the trivialization that satisfies the given requirements on boundary. Then we just glue $\Psi_1$ and $\Psi_U \circ \Psi_2$ to obtain the global trivialization $\Phi$. $\square$

As a Corollary, we know that

**3.33 Corollary.** On a closed unit disc $\mathbb{D}$, any bundle pair $(E, F)$ with $E$ a line bundle is isomorphic to the trivial bundle $\mathbb{C} \times \mathbb{D}$ over $\mathbb{D}$ with the boundary given by $e^{ik\theta/2}\mathbb{R}$ for some $k$.

The positive integer $k$ is exactly the Maslov index of the loop $\mathbb{S}^1 \to R(n)$. In the case when $\Sigma = \mathbb{D}$, we know that the boundary Maslov index is well-defined: any bundle isomorphism $U\colon (\underline{\mathbb{C}}^n, F_1) \to (\underline{\mathbb{C}}^n, F_2)$, restricted to the boundary, gives a transformation of the loops $\Lambda_2 = U\Lambda_1$, and this isomorphism is extended naturally to the whole disc, and is hence contractible, which implies $\mu(\Lambda_2) = \mu(\Lambda_1)$.

The proof of theorem 3.29 relies on the **pair of pants induction**. That is, for any compact Riemann surface $\Sigma$ possibly with boundary, there exists a pair of pants decomposition as we stated in appendix B. The pair of pants induction is the procedure that once we have shown a theorem is true for both the disc $\mathbb{D}$ and a pair of pants $\mathbb{P}$, and we show that if we have a decomposition of Riemann surfaces $\Sigma_{02} = \Sigma_{01} \cup \Sigma_{12}$ and this theorem holds for both $\Sigma_{12}$ and $\Sigma_{01}$, then this theorem holds for $\Sigma_{02}$. From the pair of pants decomposition, it follows directly that if we complete all these three steps, then the theorem holds for all compact Riemann surfaces.

*Proof of Theorem 3.29.* The uniqueness follows from firstly the normalization axiom tells us the Maslov index defined on the line bundle over $\mathbb{D}$ is unique, and direct sum axiom and Proposition 3.32 tells us this then holds for all bundle pairs $(E, F)$ over $\mathbb{D}$. Now for a general compact Riemann surface $\Sigma$, we apply the pair of pants induction and observe that a pair of pants is a closed unit disk $\mathbb{D}$ subtracted by two open disks, and from the composition axiom and the decomposition $\mathbb{D} = \mathbb{D}_1 \cup (\mathbb{D} \setminus \operatorname{Int} \mathbb{D}_1)$ where $\mathbb{D}_1 \subseteq \mathbb{D}$ is a subdisk of $\mathbb{D}$, we know that the boundary Maslov index is defined uniquely on $\mathbb{D} \setminus \operatorname{Int} \mathbb{D}_1$ and then on a pair of pants $\mathbb{P}$. Now we just go by a pair of pants induction.

For existence, since we have shown that the construction of boundary Maslov index on the closed unit disk $\mathbb{D}$ is independent of the choice of trivializations, it suffices to verify the axioms for the boundary Maslov index. The direct sum axiom, the isomorphism axiom and the normalization are obvious, so it suffices to verify the composition axiom. If we have decomposed a Riemann surface $\Sigma_{02}$ into two parts $\Sigma_{01} \cup \Sigma_{12}$, then we know that $\Sigma_{01} \cap \Sigma_{02}$ has reverse orientations as boundaries of $\Sigma_{01}$ and of $\Sigma_{02}$, hence the sum of their degree cancels. On the other hand, components of $\partial\Sigma_{02}$ has the same orientation in $\Sigma_{02}$ and in $\Sigma_{01}$ and $\Sigma_{12}$, thus it follows that $\mu(E_{02}, F_0 \cup F_2) = \mu(E_{01}, F_0 \cup F_1) + \mu(E_{12}, F_1 \cup F_2)$. This proves existence in the case $\partial\Sigma_{02} \neq \emptyset$. For existence in the case when $\partial\Sigma_{02} = \emptyset$, we need some more argument, and this is done in the proof below. $\square$

*Proof of Theorem 3.30.* For the case $\partial\Sigma_{02} = \emptyset$, the composition gives two compact Riemann surfaces with boundary, hence we could start with the boundary Maslov indices of the decompositions. For convenience of notations, let's write $\Sigma = \Sigma_{02}$, $E = E_{02}$, $F = F_1$, $\Sigma_0 = \Sigma_{01}$ and $\Sigma_1 = \Sigma_{12}$ with the corresponding decompositions of $E$ into $E_0$ and $E_1$. Then we already have the boundary Maslov index $\mu(E_0, F)$ and $\mu(E_1, F)$. From theorem 3.32 we could further assume $E$ is a line bundle and there are trivializations $\Phi_i \colon E_i \xrightarrow{\simeq} \Sigma_i \times \mathbb{C}$. Pick an hermitian inner product $h$ on $\Sigma$, then we can assume $\Phi_i$ to be unitary and there exists a global section $s_i$ of $\mathbb{C} \times \Sigma_i$ with norm 1 over $\Sigma_i$ such that $s_i|_{\partial\Sigma_i}$ generates the totally real subbundle $\Phi_i|_{\partial\Sigma_i}(F)$. We could then find a unitary map $u \colon \partial\Sigma_0 \to \mathrm{U}(1)$ so that $s_1(z) = u(z)s_0(z)$, and by definition of Maslov index, we have

$$\mu(s_1) - \mu(s_2) = 2 \deg u.$$

Therefore it suffices to compute the degree of $u$. Write $\Gamma$ for the boundary $\partial\Sigma_i$. Now pick an hermitian connection $\nabla$ with respect to $h$, and we define complexvalued 1-forms

$$\alpha_0 = s_0^{-1}\nabla s_0, \quad \alpha_1 = s_1^{-1}\nabla s_1.$$

By a direct calculation, we have the curvature $F^\nabla = \mathrm{d}\alpha_0$ on the trivialization of $\Sigma_0$, $F^\nabla = \mathrm{d}\alpha_1$ on the trivialization of $\Sigma_1$, and $(\alpha_1|_\Gamma - \alpha_0|_\Gamma) = u^{-1}\mathrm{d}u$. Integration over $\Gamma$ gives

$$\begin{aligned}\deg u &= \frac{1}{2\pi i}\int_\Gamma u^{-1}\mathrm{d}u = \frac{i}{2\pi}\int_\Gamma(\alpha_0 - \alpha_1) = \frac{i}{2\pi}\left(\int_\Gamma \alpha_0 - \int_\Gamma \alpha_1\right)\\ &= \frac{i}{2\pi}\left(\int_{\Sigma_0}\mathrm{d}\alpha_0 + \int_{\Sigma_1}\mathrm{d}\alpha_1\right) = \frac{i}{2\pi}\int_\Sigma F^\nabla\\ &= \langle c_1(E), [\Sigma]\rangle\end{aligned}$$

where the last identity follows from the famous Chern-Weil theory(see for example, chapter 5 in [Mor01]). Therefore we have $\mu(E_0, F) + \mu(E_1, F) = 2\langle c_1(E), [\Sigma]\rangle$, as required. □

*Proof of Corollary 3.31.* Without loss of generality, we assume that $(E_i, F_i)$ are bundle pairs of rank 1, and firstly assume in particular $\Sigma$ has only one boundary component, then on the boundary $\mathbb{S}^1$ we have a smooth map $u\colon \mathbb{S}^1 \to \mathbb{S}^1$ such that $\Lambda_2(z) = u(z)\Lambda_1(z)$, where we pick trivializations of both $(E_1, F_1)$ and $(E_2, F_2)$ and let $\Lambda_i$ be the corresponding path with values in $R(n)$ of the same Maslov index, then $u$ has degree 0, hence we can realize a homotopy from $u$ to 1 on a collar neighbourhood of $\partial\Sigma$, and extend it by 1 to the whole space $\Sigma$.

In general, if $\deg u_C = 0$ for some boundary component $C$, then we can glue a disk to $C$, then $\Sigma$ would have one boundary component less, and if $\deg u_{C_i} \neq 0$ for two different boundary components $C_1$ and $C_2$(This should be the case since if not, we cannot expect the same boundary Maslov index of these two bundles), then we can pick two points on $C_1$ and $C_2$ such that $u$ takes the same value, connect them through a path in $\Sigma$, and extend $u$ to be the constant map along this path, cut $\Sigma$ along this path to obtain a new Riemann surface with one boudary component less. By this procedure we can finally reduce to the case when $\Sigma$ has only one boundary components, and therefore we can extend $u$ to the whole $\Sigma$ and obtain a global isomorphism $(E_1, F_1) \simeq (E_2, F_2)$.

The remaining case to treat with is when $\Sigma$ has no boundary components, but in this case, this follows directly from the classification of line bundles over any paracompact topological space, that the line bundle is completely classified by the first Chern class.([MS74]) □

Using the boundary Maslov index, we can now formulate the main theorem of this paragraph. For general $l \in \mathbb{Z}_{>0}$ and $p > 1$, we write $W^{l,p}_F(\Sigma, E)$ for the Sobolev space of sections $s$ on $\Sigma$ with values in $E$ satisfying the boundary conditions $s|_{\partial\Sigma} \in W^{1,p}(\Sigma, F)$.

**3.34 Theorem** (Riemann-Roch)**.** Let $D_F$ be a real linear Cauchy-Riemann operator on a complex vector bundle $E$ over a compact Riemann surface $\Sigma$, possibly with boundary, and let $lp > 2$ where $l \in \mathbb{Z}_{>0}$ and $p > 1$. Then for any $\mathbb{Z}_{>0} \ni k \leq l$ and $q > 1$ such that $k - \dfrac{2}{q} \leq l - \dfrac{2}{p}$, the map

$$D_F\colon W^{k,q}_F(\Sigma, E) \to W^{k-1,q}(\Sigma, \Omega^{0,1}(E))$$

is Fredholm, and its kernel is independent of the choice of almost complex structures and the numbers $k, q$. This holds also for the dual operator

$$D_F^*: W_F^{k,q}(\Sigma, \Omega^{0,1}(E)) \to W^{k-1,q}(\Sigma, E).$$

The Fredholm index of $D_F$ is given by

$$\operatorname{Ind}(D_F) = n\chi(\Sigma) + \mu(E, F)$$

where $\chi(\Sigma)$ is the euler characteristic of $\Sigma$ and $n$ is the rank of $E$.

The Fredholm theory for complex and smooth linear Cauchy-Riemann equations is nothing new: it is even easier than what we have done for pseudo-holomorphic strips: in that case we are treating with non-compact Riemann manifolds, so we must concern the asymptotic behaviour of such an operator, and apply some methods in ordinary differential equations. However, in this case the Fredholm theory follows directly from the elliptic estimate A.11. The general non-smooth case is much harder, and it requires some spaces to give a proof. This is because, any real linear C-R operator of class $W^{l,p}$ has the form $D_F = D + A$ where $A$ is some $W^{l-1,q}$-section of the tensor bundle $T^*\Sigma \otimes_{\mathbb{R}} \operatorname{End}_{\mathbb{R}}(E)$, and it does not follow directly that the product of two $L^q$-functions are again $L^q$, so in this case we need some further expositions. The first thing is to remark that

**3.35 Proposition.** For $l, k, p, q$ given above and $A \in W^{l,p}(\Sigma, T^*\Sigma \otimes_{\mathbb{R}} \operatorname{End}_{\mathbb{R}}(E))$, the linear opeartor

$$W^{k,q}(\Sigma, E) \to W^{k-1,q}(\Sigma, T^{0,1}\Sigma \otimes E) \colon \xi \mapsto (A\xi)^{0,1}$$

is well-defined and compact.

*Proof.* We apply the Sobolev embedding theorem. If $k = l$ then $q \leq p$, and by Hölder's inequality, $W^{l-1,p} \hookrightarrow W^{k-1,q}$. Since the map $W^{1,p}(K) \to L^p(K)$ given by multiplying a $L^p$-function is well-defined(from the Sobolev embedding theorem, the Hölder's inequality, and the assumption that $K$ be compact), the first statement holds for $k = l$. For $k \leq l-2$, the relation $k - \frac{2}{q} \leq l - \frac{2}{p}$ tells us that we have the embedding $W^{l-1,q}(\Sigma, T^*\Sigma \otimes_{\mathbb{R}} \operatorname{End}_{\mathbb{R}}(E)) \hookrightarrow C^{k-1}(\Sigma, T^*\Sigma \otimes_{\mathbb{R}} \operatorname{End}_{\mathbb{R}}(E))$, and the result follows directly. For $k = l-1$, we have $\dfrac{2}{p} \leq 1 + \dfrac{2}{q} \Rightarrow q \leq \dfrac{2p}{2-p}$, hence from Sobolev embedding theorem, we have $W^{l-1,p}(\Sigma, T^*\Sigma \otimes_{\mathbb{R}} \operatorname{End}_{\mathbb{R}}(E)) \hookrightarrow W^{k-1,q}(\Sigma, T^*\Sigma \otimes_{\mathbb{R}} \operatorname{End}_{\mathbb{R}}(E))$, which gives the required result.

Now for the compactness, we just apply the Rellich-Kondrachov theorem. If $kq > 2$ and $q > 2$ then we have the compact embedding $W^{k,q} \hookrightarrow C^{k-1}$ and hence the operator is obviously compact. If $q \leq 2$, then we only have compact embeddings $W^{k,q} \hookrightarrow W^{k-1,r}$ for any $q \leq r < \frac{2q}{2-q}$, but since $\Sigma$ is compact, some of them embeds into $W^{k-1,q'}$ where $q'$ is conjugate to $q$, and the compactness follows.

If $kq \leq 2$, then we must have $k = 1$ and $q \leq 2$. In this case, we just use the Rellich embedding for $W^{1,q} \hookrightarrow L^r$ for any $q \leq r < \frac{2q}{2-q}$, and since $q \leq 2$, it follows that $W^{1,q} \hookrightarrow L^{q'}$ is compact where $q'$ is conjugate to $q$, therefore in any cases, this operator is compact(and hence bounded). $\square$

Our main result of the Fredholm theory is the following

**3.36 Theorem.** Let $D$ be a real linear Cauchy-Riemann operator of class $W^{l-1,p}$, and $l, k, p, q$ given as in theorem 3.34. Let $r > 1$ such that $\frac{2}{r} - 1 \leq k - \frac{2}{q}$. Then the following holds:

i) The operators $D_F$ and $D_F^*$ are Fredholm and $\mathrm{Ind}(D_F) + \mathrm{Ind}(D_F^*) = 0$. Moreover, the Fredholm index of $D_F$ is independent of the choice of $D$ and of the complex structure on $\Sigma$. It is also independent of $k$ and $q$.

ii) If $\eta \in L^r(\Sigma, T^{0,1}\Sigma \otimes E)$ and $\xi \in W^{k-1,q}(\Sigma, E)$ satisfy

$$\int_\Sigma \langle \eta, D_F\zeta \rangle \mathrm{d}\,\mathrm{vol} = \int_\Sigma \langle \xi, \zeta \rangle \mathrm{d}\,\mathrm{vol} \tag{10}$$

for all $\zeta \in W_F^{k,q}(\Sigma, E)$, then $\eta \in W_F^{k,q}(\Sigma, T^{0,1}\Sigma \otimes E)$ and $D_F^*\eta = \xi$.

iii) If $\xi \in L^r(\Sigma, E)$ and $\eta \in W^{k-1,q}(\Sigma, T^{0,1}\Sigma \otimes E)$ satisfy

$$\int_\Sigma \langle D_F^*\zeta, \xi \rangle \mathrm{d}\,\mathrm{vol} = \int_\Sigma \langle \zeta, \eta \rangle \mathrm{d}\,\mathrm{vol}$$

for all $\zeta \in W_F^{k,q}(\Sigma, T^{0,1}\Sigma \otimes E)$, then $\xi \in W_F^{k,q}(\Sigma, E)$ and $D_F\xi = \eta$.

This is just a restatement of the first part of theorem 3.34. Note that the second and third statements tell us that $\mathrm{Im}D_F \perp \ker D_F^*$ and $\mathrm{Im}D_F^* \perp \ker D_F$, so once we prove the Fredholm property for $D_F$, it follows directly the Fredholm property of $D_F^*$ as well as the identity concerning Fredholm index. We begin with a lemma concerning embeddings of function spaces.

**3.37 Lemma.** *1. Set $s > 1$ be conjugate to $r$, then there are inclusions $W^{k-1,q} \hookrightarrow L^s$ and $L^r \hookrightarrow (W^{k-1,q})^*$.*

*2. There is an inclusion $W^{k,q} \hookrightarrow (W^{l-1,p})^*$.*

*3. There is an inclusion $W^{k,q} \hookrightarrow (W^{k-1,q})^*$ if and only if $-\frac{1}{2} \leq k - \frac{2}{q}$.*

*Proof.* We only need to verify the case when $(k-1)q \leq 2)$. In this case, note that we have $\frac{2}{r} - 2 \leq k - 1 - \frac{2}{q}$ and $\frac{2}{s} + \frac{2}{r} = 2$, hence $\frac{2}{q} + 1 - k \leq \frac{2}{s}$ and therefore $s \leq \frac{2q}{2-(k-1)q}$. Then from Hölder's inequality, we have $W^{k-1,q} \hookrightarrow L^s$. By taking dual we get the natural map $L^r \to (W^{k-1,q})^*$, but then using the fact that $W^{k-1,q}$ is dense in $L^s$, it follows that the natural map is an inclusion, hence an embedding.

For the second one, it suffices for us to show for the case when both $kq < 2$ and $(l-1)p < 2$. In this case, we have $W^{l-1,p} \hookrightarrow L^{\frac{2p}{2-lp+p}}$ and hence $L^{\frac{2p}{(l+1)p-2}} \hookrightarrow (W^{l-1,p})^*$. Since $lp > 2$, it follows that $\frac{2}{q} - k \leq l + 1 - \frac{2}{p}$, hence we have the embedding $L^{\frac{2q}{2-kq}} \hookrightarrow L^{\frac{2p}{(l+1)p-2}}$. Then the second statement follows.

For the last statement, we only consider the case $kq < 2$. In this case, we have $k = 1$ and $q < 2$, so we have Sobolev embeddings $W^{1,q} \hookrightarrow L^{\frac{2q}{2-q}}$ and $W^{k-1,q} = L^q$. Then $L^{\frac{2q}{2-q}}$ embeds into $L^{\frac{q}{q-1}}$ if and only if $-\frac{1}{2} \leq 1 - \frac{2}{q} = k - \frac{2}{q}$. $\square$

This tells us in what cases the integration $\int_\Sigma \langle \eta, \xi \rangle \mathrm{d}\,\mathrm{vol}$ is well-defined.

*Proof in the case $D_F$ is smooth.* Recall that we have the following elliptic estimate for $\bar{\partial}$:

$$\|u\|_{W^{1,p}(V)} \leq C(\|\bar{\partial}u\|_{L^p(U)} + \|u\|_{L^p(U)})$$

where $V \subseteq U \subseteq \mathbb{H}$ are open subsets such that $\bar{V} \subseteq U$. This inequality glues to the global inequality

$$\|\xi\|_{W^{1,p}(\Sigma,G)} \leq C(\|D\xi\|_{L^p(\Sigma,G)} + \|\xi\|_{L^p(\Sigma,G)})$$

for any vector bundle $G \to \Sigma$. Hence we know that $D$ has finite-dimensional kernel and closed image. Therefore it suffices for us to prove statements 2 and 3 and the direct sum decompositions $W^{k,q}(\Sigma, T^{0,1}\Sigma \otimes E) \cong \mathrm{Im} D_F \oplus \ker D_F^*$ and $W^{k,q}(\Sigma, E) \cong \mathrm{Im} D_F^* \oplus \ker D_F$.

Let $\eta, \xi$ and $\zeta$ as in statement 2. We can pick bump functions so that the question is local, and locally with a chosen unitary frame for $E$, we can assume that $E = U \times \mathbb{C}^n$ with the standard almost complex structure $i = J$, the hermitian connection reads $\nabla_s = \partial_s + \Phi$ and $\nabla_t = \partial_t + \Psi$, and the volume form is given by $\mathrm{d}\,\mathrm{vol} = \lambda^2 \mathrm{d}s \wedge \mathrm{d}t$. Then the identity (10) reads

$$\int_U \langle \eta, \partial_s \zeta + i\partial_t \zeta \rangle \mathrm{d}s\mathrm{d}t = \int_U \langle \lambda^2 \xi - \Phi^* \eta + \Psi^* i\eta, \zeta \rangle \mathrm{d}s\mathrm{d}t,$$

where $\Phi^*$ and $\Psi^*$ are real adjoints of these operators. Here $\eta \in L^r$, $\Phi$ and $\Psi$ are smooth, and $\xi \in W^{k-1,q}$, hence $\lambda^2\xi - \Phi^*\eta + \Psi^* i\eta \in L^s$ where $1 < s = \min\{r, q\}$. In order to obtain regularity for $\eta$, observe that the above identity reads $(\eta, \bar{\partial}\zeta)_{L^2} = (L\eta, \zeta)_{L^2}$ for some operator $L$, and by further applying $\partial$ we get $\Delta\eta = -\partial L\eta$ in the weak sense, hence applying Proposition A.12 we obtain

$$\|\eta\|_{W^{1,s}(V)} \leq C(\|L\eta\|_{L^s(U)} + \|\eta\|_{L^s(U)})$$

here we use the fact that $\Sigma$ is compact, hence locally we must have the embeddings $L^r \hookrightarrow L^s$ whenever $s \leq r$. This implies that $\eta \in W^{1,s}_{loc}(U)$ and for the boundary condition $\eta(\mathbb{R} \times \{0\}) \subseteq \mathbb{R}^n \times \{0\}$, we pick any smooth sequence of functions $v_i$ that converges in $W^{1,s}$ to $\eta$(here we extend $U$ to the double region $\tilde{U}$ and consider the natural extension of $\eta$ to the larger region), and let

$$\eta_i(s,t) = \frac{1}{2}(v_i(s,t) + Rv_i(s,-t))$$

be another sequence of smooth functions, where $R$ is the matrix such that $\eta(s,-t) = R\eta(s,t)$ for $t > 0$. Now we know that $\eta_i$ satisfies the given boundary condition and using the trace theorem(see for example, section 5.4 of [Eva10]), we conclude that $\eta_i$ converges in $L^s$ on the boundary $U \cap (\mathbb{R} \times \{0\})$ to $\eta$, and therefore $\eta(s,0) \in \mathbb{R}^n \times \{0\}$ for a.e. $s$. This proves the boundary condition of $\eta$, and therefore $\eta \in W^{1,s}_{loc,F}(U, \mathbb{R}^{2n})$. If $s \geq 2$, then $\eta \in L^q_{loc}(U, \mathbb{R}^{2n})$, and if $s < 2$, we have $\eta \in L^t(U, \mathbb{R}^{2n})$ where $s \leq t < \frac{2s}{2-s}$. In this case, $\lambda^2\xi - \Phi^*\eta + \Psi^* i\eta \in L^{r'}_{loc}$ where $r' = \min\{q, \frac{2s}{2-s}\}$, so either $r' = q$ or $r' = \frac{2s}{2-s} > 2$ since $s > 1$. In the latter case, we have $\eta \in W^{1,2}_{loc}$ and therefore in any cases $\eta \in L^q_{loc}$. Now we apply again the elliptic estimate to conclude that $\eta \in W^{1,q}_{loc}$. We could then iterate the process to obtain that $\eta \in W^{k,q}_{loc}$ and since $\Sigma$ is compact, we glue to obtain $\eta \in W^{k,q}_F(\Sigma, T^{1,0}\Sigma \otimes E)$.

Then $\eta$ is weakly differentiable and $D_F^*\eta = \xi$. The estimate for the third one is almost the same as the second statement, and we omit the proof.

Now we prove statement 1. It suffices for us to show the direct sum decomposition. If $\eta \in \mathrm{Im}D_f \cap \ker D_F^*$, then we have $D_F^*\eta = 0$ and hence by elliptic bootstrapping it follows that $\eta$ is smooth, and identity (10) tells us for any $\zeta \in W_F^{k,q}(\Sigma, E)$, we have $(\eta, D_F\zeta)_{L^2} = 0$. Therefore $\eta$ belongs to the annihilator of the subspace $\mathrm{Im}D_F \subseteq L^q(\Sigma, T^{0,1}\Sigma \otimes E)$, but note that $\eta$ is always smooth, hence $\eta \in L^q(\Sigma, T^{0,1}\Sigma \otimes E)$ as well, but then we could pick $\zeta$ to be one of the preimage of $\eta$ and we have $(\eta, D_F\zeta)_{L^2} = (\eta, \eta)_{L^2} = 0$. Therefore $\eta = 0$. Let $\zeta \in L^r$ be a section such that $\zeta \perp (\mathrm{Im}D_F \oplus \ker D_F^*)$, then for any $\xi \in W_F^{1,q}$ we have $(\zeta, D_F\xi)_{L^2} = 0$ and hence $\xi \in \ker D_F^* \subseteq L^r$. However, $\xi \perp \ker D_F^*$, so we have $\xi = 0$ and therefore by Hahn-Banach theorem and the density of $L^r$ in $L^{q'}$, we conclude that $L^q = \mathrm{Im}D_F \oplus \ker D_F^*$. The decomposition for $\mathrm{Im}D_F^*$ and $\ker D_F$ is similar. This proves the theorem in the case $k = 1$, and for general $k$, it's obvious that the intersection is still trivial, and for the decomposition, let $\zeta \in L^r \hookrightarrow (W^{k-1,q})^*$ be the function such that $\zeta \perp \mathrm{Im}D_F \oplus \ker D_F^*$, from $\zeta \perp \mathrm{Im}D_F$ we know that $\zeta \in W^{k,q}$ and that $\zeta \in \ker D_F^*$, so $\zeta \perp \ker D_F^*$ implies that $\zeta = 0$. Therefore from density the decomposition for general $k$ follows.

Finally, we want to show that the Fredholm index of $D_F$ is independent of the choice of $D$ and the complex structure on $\Sigma$, but the first one just follows directly from the fact that the difference of any two such C-R operators is a compact operator from $W_F^{k,q}$ to $W^{k-1,q}$, and for the second one, given any smooth path $\lambda \mapsto j_\lambda$ of compatible almost complex structures with $0 \leq \lambda \leq 1$, we could modify $\lambda$ such that near 0 and 1 the almost complex structures are the same and construct the product manifold $I \times \Sigma$ where $I$ is the open unit interval and consider the pull-back of $T\Sigma$ to $I \times \Sigma$. We could then construct an induced almost complex structure $j$ on $I \times \Sigma$ such that for each $t \in I$, $j|_{t \times M} = j_t$. Then we just choose a complex linear connection $\nabla$ on $p^*T\Sigma \to I \times M$ and consider the parallel transport by this connection. This would give the complex linear bundle isomorphism $\Phi_t$ of $T\Sigma$. Now we just pick an hermitian connection $\nabla$ on $E$ and consider the corresponding family of Cauchy-Riemann operators $D_\lambda$. The composition $D_\lambda \circ \Phi_\lambda$ is again Fredholm, and from results in functional analysis(c.f. [RS01]), we know that the Fredholm index of this family is the same. Therefore the Fredholm index is independent of the choice of almost complex structures. □

Now we must consider the case when $D_F$ is no longer smooth but only of class $W^{l,p}$ for $lp > 2$, and we have $k \leq l$ and $k - \frac{2}{q} \leq l - \frac{2}{p}$. We first prove part one.

*Proof of Statement (i) in General.* From Proposition 3.35 we know that the non-smooth zero-order term $A$ is a compact operator viewed as a linear operator from $W_F^{k,q} \to W^{k-1,q}$, hence the Fredholm index of $D_F$ is the same as some corresponding smooth Cauchy-Riemann operator, which is independent of the choice of $k, q$ and the almost complex structure of $T\Sigma$, so is $D_F$. The statement for $D_F^*$ is also the same. □

From Lemma 3.37 we know that we have the inclusion $W^{k,q} \hookrightarrow (W^{k-1,q})^*$ if and only if we have the inequality $-\frac{1}{2} \leq k - \frac{2}{q}$, so we firstly consider the case when $(k, q)$ satisfies this given inequality.

*Proof of (ii) and (iii) in the case $-\frac{1}{2} \le k - \frac{2}{q}$.* We firstly show that the decomposition

$$W^{k-1,q}(\Sigma, T^{0,1}\Sigma \otimes E) \cong \mathrm{Im} D_F \oplus \ker D_F^*$$

still holds. In the non-smooth case, we also have the elliptic estimate for $D_F$ and $D_F^*$, so any $\eta \in \ker D_F$ and $\zeta \in \ker D_F^*$ are of class $W^{l,p}$, then by a direct calculation:

**3.38 Lemma.** *For any $\xi \in \Omega^0_F(\Sigma, E)$ and $\eta \in \Omega^{0,1}_F(\Sigma, E)$, we have*

$$\int_\Sigma \langle \eta, D_F \xi \rangle \mathrm{d}\,\mathrm{vol} = \int_\Sigma \langle D_F^* \eta, \xi \rangle \mathrm{d}\,\mathrm{vol}.$$

and the natural inclusion $W^{k,q} \to (W^{k-1,q})^*$, we know that there is a natural map $\ker D_F \to \mathrm{coker}\, D_F^*$ and conversely $\ker D_F^* \to \mathrm{coker}\, D_F$. They are inclusions, and the fact that $\mathrm{Ind}(D_F) + \mathrm{Ind}(D_F^*) = 0$ gives us the identity

$$\dim \ker D_F + \dim \ker D_F^* = \dim \mathrm{coker}\, D_F + \dim \mathrm{coker}\, D_F^*,$$

hence we know that the map $\ker D_F \oplus \ker D_F^* \to \mathrm{coker}\, D_F \oplus \mathrm{coker}\, D_F^*$ is an isomorphism, giving the required isomorphism $\ker D_F \cong \mathrm{coker}\, D_F^*$ and $\ker D_F^* \cong \mathrm{coker}\, D_F$. Therefore the decomposition

$$W^{k-1,q}(\Sigma, E \otimes T^{0,1}\Sigma) \cong \mathrm{Im} D_F \oplus \ker D_F^*$$

and the similar result for $W^{k-1,q}(\Sigma, E)$ continues to hold in the general case.

Now we just try to prove the statement. Let $\eta \in L^r(\Sigma, T^{0,1}\Sigma \otimes E)$ and $\xi \in W^{k-1,q}(\Sigma, E)$ satisfy the given identity, then we have the decomposition $\xi = \xi_0 + D_F^* \eta_0$ where $\eta_0 \in W^{k,q}_F$ and $\xi_0 \in \ker D_F \subseteq W^{k-1,q}(\Sigma, E)$, so the identity reads

$$(\eta, D_F \zeta)_{L^2} = (\xi, \zeta)_{L^2} = (\xi_0, \zeta)_{L^2} + (D_F^* \eta_0, \zeta)_{L^2} = (\xi_0, \zeta)_{L^2} + (\eta_0, D_F \zeta)_{L^2}.$$

Since $D_F \xi_0 = 0$ and hence $\xi_0 \in W^{k,q}(\Sigma, E)$, we could pick $\zeta = \xi_0$ and obtain

$$\|\xi_0\|^2_{L^2} = (\xi_0, \xi_0)_{L^2} = (\eta - \eta_0, D_F \xi_0)_{L^2} = 0.$$

Therefore $\xi_0 = 0$, hence we have $(\eta - \eta_0, D_F \zeta)_{L^2} = 0$ for all $\zeta \in W^{k,q}(\Sigma, E)$. This implies that $\eta - \eta_0 \in \mathrm{coker}\, D_F \cong \ker D_F^*$, hence if we write $\eta_1 = \eta - \eta_0$, then we have $\eta_1 \in \ker D_F^* \subseteq W^{k,q}_F(\Sigma, E \otimes T^{0,1}\Sigma)$ and $D_F^* \eta_1 = 0$. Then we know that $\eta \in W^{k,q}_F(\Sigma, E \otimes T^{0,1}\Sigma)$ and $D_F^* \eta = D_F^* \eta_0 = \xi$. This gives the required result. □

Finally, we drop the requirement that $-\frac{1}{2} \le k - \frac{2}{q}$. This relies on the following elliptic estimate:

**3.39 Proposition.** For every $\xi \in W^{k,q}_F(\Sigma, E)$, $D_F \xi = 0$ implies that $\xi \in W^{l,p}_F(\Sigma, E)$. Similar results hold for $D_F^*$. Moreover, the inclusions $\ker D_F \hookrightarrow \mathrm{coker}\, D_F^*$ and $\ker D_F^* \hookrightarrow \mathrm{coker}\, D_F$ are isomorphisms and the decompositions

$$W^{k-1,q}(\Sigma, E) \cong \ker D_F \oplus \mathrm{Im} D_F^*, \tag{11}$$

$$W^{k-1,q}(\Sigma, T^{0,1}\Sigma \otimes E) \cong \ker D_F^* \oplus \mathrm{Im} D_F \tag{12}$$

hold.

*Proof.* Since $lp > 2$, we have the inclusion $W^{l,p} \hookrightarrow (W^{k-1,q})^*$, hence we have the natural inclusion of $\ker D_F$ into $\operatorname{coker} D_F^*$ and similarly $\ker D_F^* \hookrightarrow \operatorname{coker} D_F$. Then we have the isomorphism $\ker D_F \cong \operatorname{coker} D_F^*$ and $\ker D_F^* \cong \operatorname{coker} D_F$. The rest then follows from this isomorphisms. We just iterate the proof above to obtain decompositions (11) and (12). □

With this decomposition, we could then follow the proof above in special cases to obtain theorem 3.36. This is completely similar and hence will be omitted. □

For the second part, we need the calculation of Fredholm index. Again we use the pair of pants induction, so firstly

**3.40 Proposition.** Assume that $\Sigma = \mathbb{D}$ and $(E, F)$ is a trivial line bundle over $\mathbb{D}$ with $F = e^{ik\theta/2}$ over $\mathbb{S}^1$ for some $k \in \mathbb{Z}$. Then theorem 3.34 holds for this bundle pair $(E, F)$ and the standard Cauchy-Riemann operator $\bar{\partial} := D_F$ over the closed unit disk $\mathbb{D}$.

*Proof.* We firstly give a characterisation of the cokernel of $D_F$, that is, the kernel of $D_F^*$. Note that for standard $\xi \in W_F^{k,p}(\mathbb{D}, \mathbb{C})$ we have

$$D_F\xi = \frac{1}{2}(\partial_s\xi + i\partial_t\xi)(\mathrm{d}s - i\mathrm{d}t)$$

and the conjugate operator is given by $D_F^*(\zeta \mathrm{d}\bar{z}) = (\partial_s - i\partial_t)\zeta$, and it suffices to determine the boundary conditions for $\zeta \in W^{k,q}(\mathbb{D}, \Omega^{0,1})$ such that $D_F^*\zeta = 0$. By definition,

$$\begin{aligned}\int_{\mathbb{D}} \langle \zeta \mathrm{d}\bar{z}, D_F\xi\rangle \mathrm{d}s\mathrm{d}t &= \operatorname{Re}\int_{\mathbb{D}} \bar{\zeta}(\partial_s\xi + i\partial_t\xi)\mathrm{d}s\mathrm{d}t + \operatorname{Re}\int_{\mathbb{D}} \overline{(\partial_s\zeta - i\partial_t\zeta}\xi \mathrm{d}s\mathrm{d}t \\ &= \operatorname{Re}\int_{\mathbb{D}} \partial_s((\bar{\zeta}\xi) + i\partial_t(\bar{\zeta}\xi))\mathrm{d}s\mathrm{d}t = \operatorname{Re}\int_0^{2\pi} e^{i\theta}\bar{\zeta}(e^{i\theta})\xi(e^{i\theta})\mathrm{d}\theta\end{aligned}$$

The last equality follows from Green's formula. Now since $\xi \in W_F^{k,p}$, it follows that over the boundary, $\xi(e^{i\theta}) \in e^{ik\theta/2}\mathbb{R}$, i.e. when multiplying with $e^{-ik\theta/2}$, we have $e^{-ik\theta/2}\xi \in \mathbb{R}$, thus the last term in the above formula is zero if and only if $e^{i\theta+ik\theta/2}\bar{\zeta} \in \mathbb{R}$, i.e. $\zeta(e^{i\theta}) \in e^{i\theta+ik\theta/2}\mathbb{R}$. This is the required boundary conditions for $\zeta$, and we still write $W_F^{k,q}$ for the space of all $\zeta$ with the given boundary conditions.

Now we compute explicitly the dimension of $\ker D_F$. Let $u$ be such a kernel, then $u$ is holomorphic inside $\mathbb{D}$ with $e^{-ik\theta}u(e^{i\theta}) = \bar{u}(e^{i\theta})$ on the boundary. Let's expand $u$ at 0 to obtain the power series

$$u(z) = \sum_{i=0}^{\infty} a_i z^i$$

where the radius of convergence is greater than or equal to 1, and on the boundary, the Fourier coefficient of $a_n$ can be computed as

$$a_n = \lim_{r\to 1}\frac{1}{2\pi}\int_{\mathbb{S}^1} \xi(re^{i\theta})e^{-in\theta}\mathrm{d}\theta = \frac{1}{2\pi}\int_{\mathbb{S}^1} \xi(e^{i\theta})e^{-in\theta}\mathrm{d}\theta$$

thus we have $a_n = \bar{a}_{k-n}$ for all $n \in \mathbb{Z}$, and since $a_n = 0$ when $n < 0$, it follows that $a_n = 0$ for $n \geq k+1$ and therefore $u$ can only be the polynomials

$$u(z) = \sum_{i=0}^{k} a_i z^i$$

with $a_i = \bar{a}_{k-i}$. This tells us $\dim D_F = k+1$ for $k \geq 0$ and is zero for $k \leq -1$.

Similarly, for the cokernel of $D_F$, i.e. the kernel of $D_F^*$, we have $\bar{\zeta}$ holomorphic and satisfies $e^{i(2+k)\theta}\bar{\zeta}(e^{i\theta}) = \zeta(e^{i\theta})$, thus by the above result, we have that if we expand $\bar{\zeta}$ as a sum of power series with coefficients $\bar{b}_n$, then $\bar{b}_n = b_{-k-2-n}$. Since we have $\bar{b}_n = 0$ when $n < 0$, it follows that

$$\zeta(z) = \sum_{i=0}^{-k-2} b_i \bar{z}^i$$

with $b_i = \bar{b}_{-k-2-i}$. Then for $k \geq -1$, $\dim \operatorname{coker} D_F = 0$ and for $k \leq -2$, $\dim \operatorname{coker} D_F = -k-1$. This proves that for any $k$, we have $\operatorname{Ind}(D_F) = k+1 = \chi(\mathbb{D}) + \mu(E,F)$. □

The next step towards the proof of the index formula is to show the composition of two Riemann surface correponds to the sum of two Fredholm indices of two Cauchy-Riemann operators. Then the general result follows directly from the pair of pants induction.

**3.41 Theorem.** Assume that we have a decomposition of Riemann surfaces $\Sigma_{02} = \Sigma_{01} \cup \Sigma_{12}$, with decompositions of bundle pairs $(E_{02}, F_0 \cup F_2) = (E_{01}, F_0 \cup F_1) \cup (E_{12}, F_1 \cup F_2)$, and we write $D_{ij}$ for the corresponding real-linear Cauchy-Riemann operators on $\Sigma_{ij}$, $X_{ij}$ for the corresponding domains of $D_{ij}$ and $Y$ the common codomain. Then we have

$$\operatorname{Ind}(D_{02}) = \operatorname{Ind}(D_{01}) + \operatorname{Ind}(D_{12}).$$

*Proof.* The rough idea is that we construct two Banach spaces $X_0$ and $X_1$ inside the direct sum $X_{01} \oplus X_{12}$ such that they are isomorphic and the operators $D_{02}$ and $D_{01} \oplus D_{12}$ can be viewed as a perturbation of some operators $D_0 \colon X_0 \to Y$ and $D_1 \colon X_1 \to Y$, hence the final step would be to show that as complexes, we have the isomorphism

$$\begin{tikzcd} 0 \arrow[r] & X_0 \arrow[r, "D_0"] \arrow[d, "\Psi_X"] & Y \arrow[r] \arrow[d, "\Psi_Y"] & 0 \\ 0 \arrow[r] & X_1 \arrow[r, "D_1"] & Y \arrow[r] & 0 \end{tikzcd}$$

which gives the desired result. First of all, let $\Gamma = \Sigma_{01} \cap \Sigma_{12}$, then we could pick a tubular neighbourhood of $\Gamma$ in $\Sigma_{02}$, which is an embedding $\phi \colon \Gamma \times [0,1] \to \Sigma_{02}$ and moreover we can modify the almost complex structure on $\Sigma_{02}$(Since almost complex structures does not influence the Fredholm index of $D_{ij}$) so that $\phi$ is pseudo-holomorphic. On this neighbourhood, we can find a trivialization $E|_U \xrightarrow{\Phi} \mathbb{C}^n \times U$ so that $\bar{\partial} \circ \Phi = \Phi \circ D_{02}$, that is, on the trivialization the Cauchy-Riemann operator is standard. This is because any two connections over the same vector bundle differs from a vector-valued differential one-form $A$, hence we can replace $D$ by $D + A$ so that $\Phi \circ (D + A) \circ \Phi^{-1} = \bar{\partial}$ and since $A$ has order 0, $D + A$ and $D$ have the same Fredholm index.

Now we write $X_{ij} = W^{k,p}_F(\Sigma_{ij}, E_{ij})$, $\Gamma_0 = \partial\Sigma_{02} \cap \Sigma_{01}$, $\Gamma_1 = \Sigma_{01} \cap \Sigma_{12}$, $\Gamma_2 = \partial\Sigma_{02} \cap \Sigma_{12}$, $Y = L^{k-1,q} \oplus L^{k-1,q}$ and define

$$X_0 = \left\{ (\xi, \eta) \in X_{01} \oplus X_{12} \middle| \begin{array}{c} \xi|_{\partial\Sigma_{01}} \in F_0 \cup F_1; \\ \eta|_{\partial\Sigma_{12}} \in F_1 \cup F_2. \end{array} \right\}, \tag{13}$$

$$X_1 = \left\{ (\xi, \eta) \in X_{01} \oplus X_{12} \middle| \begin{array}{c} \xi(\Gamma_0) \subset F_0,\ \eta(\Gamma_2) \subset F_2; \\ \xi|_{\Gamma_1} = \eta|_{\Gamma_1}. \end{array} \right\}. \tag{14}$$

It follows that $X_1 \subseteq X_{02}$ and in fact, since we could glue these two functions up, we have $X_1 = X_{02}$. Let $D_0$ and $D_1$ be the induced operator of $D_{02}$ to these two spaces, then by definition, we know that $D_0$ is the direct sum of $D_{01}$ and $D_{12}$, thus we have

$$\mathrm{Ind}(D_0) = \mathrm{Ind}(D_{01}) + \mathrm{Ind}(D_{12}); \quad \mathrm{Ind}(D_1) = \mathrm{Ind}(D_{02}).$$

It suffices to construct an isomorphism from the sequence $0 \to X_0 \xrightarrow{D_0} Y \to 0$ to the sequence $0 \to X_1 \xrightarrow{D_1}\to Y \to 0$. This is given via homotopy. Note that over $\Gamma_1$ the space $X_1$ can be viewed as the space of sections with conditions $(\xi(z), \eta(z)) \in \Delta$ where $\Delta \subset \mathbb{C}^n \oplus \mathbb{C}^n$ is the diagonal, and in this sense, if we could construct a bundle isomorphism on $\mathbb{C}^n \oplus \mathbb{C}^n$ such that it sends the direct sum $F_1 \oplus F_1$ on the set $\Gamma_1$ to $\Delta$, then we could construct a map from $X_0$ to $X_1$. In fact,

**3.42 Lemma.** *On any connected component of $\Gamma_1$, there exists a smooth section $\Psi\colon [0,1] \times \mathbb{S}^1 \to \mathrm{GL}_{\mathbb{R}}(\mathbb{C}^n \oplus \mathbb{C}^n)$ such that $\Psi(s,t) \equiv \mathrm{id}$ for $s \geq \frac{1}{2}$, $\Psi$ commutes with the almost complex structure $I = \begin{pmatrix} -i & \\ & i \end{pmatrix}$ and $\Psi(0,t)^{-1}(\Delta) = \Lambda(t) \oplus \Lambda(t)$. Here $\Lambda(t) = \Phi(\phi(s,t))F_{1,t}$ with $t \in \mathbb{S}^1$.*

The proof is direct: with the given almost complex structure $I$, $\Lambda(t) \oplus \Lambda(t)$ has Maslov index zero, hence contractible to the diagonal $\Delta$. Then we just realize this diagonal as the section $\Psi$. What remains is the construction of $\Psi_X$ and $\Psi_Y$, but they are given directly by applying $\Psi$ to the image of the pair $(\xi, \eta)$ under the local trivialization $\phi$. Finally, by some calculation, we have $D_1 \circ \Psi_X - \Psi_Y \circ D_0$ differs from a compact operator from $X_1$ to $Y$, hence $\mathrm{Ind}(D_1) = \mathrm{Ind}(D_0)$, proving the theorem. ☐

**The Maslov-Viterbo Index** Now we apply Riemann-Roch theorem to compute the dimension of the moduli space $\mathcal{M}(x, y)$ for given critical points $x$ and $y$. Although the strip $\mathbb{S}$ is not a compact Riemann surface, when we add critical points to the strip, we will obtain a compactification which is a closed unit disk. Let $L_0$ and $L_1$ be transversal Lagrangian submanifolds and $\{J_t\}_{0\leq t\leq 1}$ a sequence of almost complex structures compactible with $\theta$ on $P$ such that $L_i$ is totally real with respect to $J_i$, then $u^*(TL_i)$ are totally real subbundles of $u^*(TP)$ when we pick a proper family of almost complex structures, and for any $t$, $u$ is pseudo-holomorphic.

**3.43 Definition.** The Maslov-Viterbo index assigns to each such $u \in \mathcal{M}(x, y; \{J_t\})$ an index $\mu(u)$ such that this is independent of the choice of $u$ but only on the choice of $x$ and $y$.

This "definition" is not quite clear. It's only a "principle" for us to construct the index, and the explicit construction of such an index is given as follows. We regard at present the strip $\mathbb{S}$ as a unit square $I \times I$, with the upper and lower boudnaries the original boundaries, and the left and right boundaries mapping to $x$ and $y$. Then on the upper and lower boundaries, we just set the totally real subbundle $F := TL_0 \cup TL_1$. On the left and right boundaries, since $T_xL_0 \neq T_xL_1$ because of transversality, for each $t$, we firstly set $\Phi_t \in \mathrm{GL}_{\mathbb{R}}(n, \mathbb{C})$ to be the linear map such that it is symplectic, $\Phi_0 = \mathrm{id}$ and $J_t\Phi_t = \Phi_t J_0$, and on the trivialization $I \times \mathbb{C}^n$ defined by $\Phi_t$, we set $\Lambda(t)$ to be the **canonical short path** from $\Lambda(0) = T_xL_0$ to $\Lambda(1) = \Phi_1^{-1}T_xL_1$.(For the concept of canonical short path, this is just what Arnold used in his paper [Arn67], the path given by rotations.) We can do the similar construction for $y$, and finally glue to a continuous family of loops on the boudnary. Then we just set $\mu(x, y) = \mu(\partial I^2)$ with the positive orientation. In this case, the loop is in fact smooth if we pick the homeomorphism from this square to the usual closed unit disk, with the left and right boudnaries of $I^2$ corresponding to a small neighbourhood of $-1$ and 1, hence we could apply the Riemann-Roch theorem 3.34 to our case and conclude that the Fredholm index of $\mathrm{d}\bar{\partial}$ is just $n + \mu(x, y)$. Since we have

**3.44 Lemma.** *For any given boundary conditions $f$ on $\mathbb{D}$ with values in a given totally real subbundle $F$ of $E$, there exists a solution $u$ satisfying $D_F u = 0$ in $\mathbb{D}$ and $u|_{\partial\mathbb{D}} = f$.*

and the exponential decay property tells us that the value of any vector field on the critical points must be 0, therefore we have $\dim \mathcal{M}(x, y) = \mu(x, y)$, proving the dimension formula for moduli spaces.

**3e) Glueing Trajectories** Although we have defined the Floer differential, we have not completely finished the construction: we need to verify that $\partial_F^2 = 0$, so that this actually gives a complex. To do this, we need a kind of Gromov compactness results. The baby version is proved by Gromov in his paper [Gro85] where he proved that the closure of the moduli space of $J$-holomorphic curves consists of "cusp-curves". In our case, we consider when $\mu(x) - \mu(y) = 2$, the boundary of the moduli space $\widehat{\mathcal{M}}(y, x)$ would be expected to consist of "brocken trajectories", as in finite-dimensional case, i.e. a pair $(u, v)$ of pseudo-holomorphic strips such that $u \in \mathcal{M}(z, x)$, $v \in \mathcal{M}(y, z)$, and $\mu(z) = \mu(x) + 1$, then we know that the closure $\overline{\widehat{\mathcal{M}}(y, x)}$ is a compact one-dimensional manifold with boundary, and from the classification of compact one-dimensional manifolds(for this, see for example chapter 5 of Lee's textbook [Lee11]), we know that the number of boundary points must be even, thus proving that $\partial_F^2$ modulo 2. Therefore we have a well-defined Floer complex $(CF(P; L_0, L_1; J; \mathbb{Z}_2), \partial_F)$. In order to avoid too much analysis details, we impose stronger conditions on the original manifold $(P, \theta)$, but the idea in the general case is the same. In fact, the compactness property is the main obstruction for symplectic topologists to proof Arnold's conjecture in a more general setting, that is when we drop the strong topological conditions and impose a restriction that $L$ be involutive. Yong-Geon Oh gave a proof in the case when $P$ is monotone in his series of papers [Oh93, Oh95, Oh06]. Further developments of compactness results were inspired from Mirror Symmetry, where Witten conjectured an invariant on the symplectic manifold given by counting holomorphic curves with prescribed marked points, which lead to the **Gromov-Witten invariant**. This was then used by two groups of people, Fukaya, Ono [FO99] and

Liu, Tian [LT98]. The first main theorem in this paragraph is that

**3.45 Theorem** (Gromov Compactness)**.** Assume that $(P, \theta)$ is a compact symplectic manifold and $L_0 \subset P$ a Lagrangian submanifold such that $\pi_2(P, L_0) = 0$. Let $L_1$ be a Lagrangian submanifold transversal to $L_0$ such that $L_1$ deforms into $L_0$ via a Hamiltonian diffeotopy. Then for any pair of intersection points $x, y \in L_0 \cap L_1$, any sequence $\{J_t\}_{0 \leq t \leq 1}$ of almost complex structures compactible with $\theta$ such that $L_i$ is totally real with respect to $J_i$, and $\{u_n\}_{n \in \mathbb{N}}$ a sequence of pseudo-holomorphic strips in $\mathcal{M}(x, y)$, there exists a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}$ and intersection points $x_1, y_1 \in L_0 \cap L_1$ such that $u_{n_k} \xrightarrow{C^\infty_{loc}} u$ for some $u \in \mathcal{M}(x_1, y_1)$ and we have the energy estimate

$$\|\nabla u\|_{L^2} \leq \limsup_{n \to \infty} \|\nabla u_n\|_{L^2}.$$

The proof is based on the Arzela-Ascoli theorem, which states that once we know that the sequence is uniformly bounded and equicontinuous, then it admits a locally uniformly convergent subsequence to some strips. In fact, the condition that the $u_n$ all lie in some space $\mathcal{M}(x, y)$ can be dropped and be replaced by a general sequence of strips $u_n \in \mathcal{P}^{k,p}_{loc}(\mathbb{S}, P)$ with the norm $E(u_n) = \frac{1}{2}\|\partial_{J_n} u_n\|_{W^{k-1,p}} \xrightarrow{n \to \infty} 0$ and the energy $E(u_n) = \frac{1}{2}\|\nabla u_n\|_{L^2} \leq C$ for some positive constant $C > 0$, where $J_n$ is a sequence of $\theta$-compactible almost complex structures that tends in $W^{k,p}_{loc}$ to some $\theta$-compactible almost complex structure $J$. This is because, if $\{u_n\} \subseteq \mathcal{M}(x, y)$, then for a chosen homotopy class $[\Gamma]$ all the strips $u_n \in \mathcal{M}(x, y)$ representing the same homotpy class has the same energy, which is the difference of the given action functionals. Then for any such sequence, we can just pick a subsequence which represents the same homotopy class to achieve this. We can also make the $L_1$ in the theorem changes, but they should all be Hamiltonian isotopic to $L_0$. In principle, when we have obtained a convergent subsequence, we could then compute the energy of this limit, and since the energy tends to 0, it follows directly that the energy of the limit of this sequence is also 0.

*Proof.* It suffices to show that for such a sequence $\{u_n\}$, and any positive integer $\rho \in \mathbb{N}$, there exists a positive constant $C_\rho$ such that

$$\|\mathrm{D} u_n\|_{L^p(\mathbb{S}_\rho)} \leq C_\rho$$

for any $n \geq 1$ and some $p > 2$, where $\mathbb{S}_\rho = [-\rho, \rho] \times [0, 1]$ is the compact subset of $\mathbb{S}$. Assume to the contrary that this does not hold, then we could find a sequence $\{\varepsilon_n\}$ tending to 0 such that $\|\mathrm{D} u_n\| \geq \varepsilon_n^{(2-p)/p}$. Explicitly, we set

$$\varepsilon_n = \inf\{\varepsilon > 0 | \|\mathrm{D} u_n\|_{L^p(B_\varepsilon(\theta_\rho))} = \varepsilon^{(2-p)/p}\},$$

then it follows that for each $\rho$, there exists a point $\theta_n \in \mathbb{S}_\rho$ such that

$$\int_{B_{\varepsilon_n}(\theta_n)} |\mathrm{D} u_n|^p \mathrm{d}\theta = \frac{1}{2}\varepsilon^{2-p}.$$

Set $B_n = B_{\varepsilon_n}(\theta_n)$. The key point is that if this divergence does hold, then near the limit of $\theta_n$ there would be a "bubbling", which gives a sphere or a disk with non-empty area, contradicting the fact that $\pi_2(P, L_0) = 0$. So we just set

$$\delta_n = \varepsilon_n^{-1} \mathrm{d}(\theta_n, \partial\mathbb{S})$$

This positive number indicates the limit point lies in the boundary or in the interior. If $\delta_n \to \infty$, then the limit would lie in the interior of $\mathbb{S}$, and if we set

$$v_n = u_n(\delta_n(\theta - \theta_n))$$

for $\theta \in \mathbb{S}$, then $\{v_n\}$ would be a sequence satisfying the following property

- $\|\nabla v_n\|_{L^2} \leq C$, since the $L^2$-norm of the gradient in independent of translation and dilation;
- $\|\nabla v_n\|_{L^p(B_1(0))} = \frac{1}{2}$, because of the rescaling and the translation;
- $\|\nabla v_n\|_{L^p(B_1(\theta))} \leq 1$ for all $\theta \in B_{\varepsilon_n^{-1}-1}(0)$, because of the minimality of $\theta_n$;
- $\|\bar{\partial} v_n\|_{L^p} \xrightarrow{n\to\infty} 0$, because of the invariance.

It follows that the sequence $\{v_n\}$ admits a subsequence tending in $W^{k,p}_{loc}(\mathbb{C}, P)$ to some pseudo-holomorphic strip $v$. This limit would satisfy $\|\nabla v\|_{L^2} \leq C$ and $\|\nabla v\|_{L^p(B_1(0))} \geq \frac{1}{2}$, but then, under the conformal map $\gamma\colon \mathbb{S}^1 \times \mathbb{R} \to \mathbb{C}^*$ the map $\tilde{v} = v \circ \gamma$ has the same $L^2$-norm on the gradient with $v$ and we have explicitly

$$\|\nabla v\|^2_{L^2} = \int_{\mathbb{R}} \int_{\mathbb{S}^1} \left( \left| \frac{\partial \tilde{v}}{\partial s} \right|^2 + \left| \frac{\partial \tilde{v}}{\partial t} \right|^2 \right) \mathrm{d}t\mathrm{d}s \geq \int_{\mathbb{R}} \|\dot{\tilde{v}}(s)\|^2_{L^2} \mathrm{d}s,$$

which implies that for any $\varepsilon > 0$ sufficiently small, there exists $s$ sufficiently large such that $\tilde{v}(s)$ is contained in an $\varepsilon$-ball in $P$, then we could extend $\tilde{v}$ to another map $\bar{v}\colon \mathbb{S}^2 \to P$(with the origin attached at one end to be a constant disk and the end points attached with a small disk in this small ball) with

$$\int \bar{v}^*\omega = \frac{1}{2}\|\nabla v\|^2_{L^2} - C_1\varepsilon^2 > 0$$

for some positive constant $C_1$ independent of $\varepsilon$. Then since $\pi_2(P, L_0) = 0$ and $L_0$ Lagrangian, it follows that we must have $\langle \omega, \pi_2(P) \rangle = 0$, hence the integral must be zero, a contradiction.

There is still another case when $\delta_n$ is bounded with limit $\delta$. Now when we apply the same procedure to construct a sequence $v_n$, the local uniform limit would then be a map $v\colon (\mathbb{H} - ir) \to P$ where $\mathbb{H}$ is the closed upper half plane, and we have $v(-ir + \mathbb{R}) \subset L$. Then we could pick a conformal equivalence $\gamma\colon \mathbb{R} \times [0,1] \to (\mathbb{H} - ir) \setminus \{-ir\}$ and consider similarly $\tilde{v} = v \circ \gamma$. The energy estimate on the gradient would imply the fact that the ends of the strip can also be attached with a disk with boundaries in $L$ so that the whole strip is made into a disk with boundaries in $L$. We then obtain a disk with non-empty area, contradicting the topological condition.

Now since we have proved the bound for any $p > 2$, for $p \leq 2$ the same bound holds using Hölder's inequality, we could apply Arzela-Ascoli theorem directly to the sequence $\{u_n\}$ to conclude the convergence. The remaining poit that $u \in \mathcal{M}(x, y)$ has been shown in section 2. $\square$

This only gives the technical part of the compactness. What we actually want is the following theorem about brocken trajectories. Here for a given real number $\tau_i$, we set $\tau_i * u_n$ to be the strip $\tau_i * u_n(s,t) := u_n(s - \tau_i, t)$, and assume $u$ is a smooth function from $\mathbb{R}$ to $\Omega(I; P, L_0, L_n)$.

**3.46 Theorem.** For any sequence $\{u_n\}$ in $\mathcal{M}_{J_n,L_n}(x-,x+)$ with convergence on $L_n$ and $J_n$, there exists a pair of finite set of points $\{\tau_{1n},\tau_{2n},\cdots,\tau_{Nn}\}\subseteq\mathbb{R}$ and $\{x_1,x_2,\cdots,x_k\}$ such that there exists a subsequence, still written $\{u_n\}$, satisfying

$$\tau_{in}*u_n\to v_i\in\mathcal{M}(x_{i-1},x_i)$$

where $x_0:=x-$ and $x_{k+1}:=x+$ in the local topology. Moreover, for any $\varepsilon>0$, there exists $n\in\mathbb{N}$ such that $v_n$ lies in the $\varepsilon$-cube

$$U_\varepsilon(v_1,\cdots,v_k)=\bigcup_{i=1}^{N}\bigcup_{\tau\in\mathbb{R}}U_\varepsilon(v_i(\tau))\subseteq\Omega(P;L_0,L),$$

where $U_\varepsilon(v_i(\tau))$ is the neighbourhood of $v_i(\tau)$ consisting of paths that with distance to $v_i$ less than $\varepsilon$.

*Proof.* We have shown in the previous technical lemma 3.45 that the sequence $u_n$ always admits a locally convergent subsequence, still written $u_n$ that converges to some pseudo-holomorphic strip $u\in\mathcal{M}(x_1-,x_1+)$. For simplicity, we give a prescribed translation of $u_n$, still written $u_n$, such that $\mathcal{A}(u_n(0),[\Gamma])=\frac{1}{2}(\mathcal{A}(x,[\Gamma])+\mathcal{A}(y,[\Gamma]))$ with the homotopy class $[\Gamma]$ prescribed. Then we have excluded the case when $x_1-=x_1+\in\{x-,x+\}$ since then the value of $\mathcal{A}$ at $u(0)$ will not coincide with $\mathcal{A}(u_n(0))$, and the case when $x_1-=x-$ and $x_1+=x+$ is easily obtained by doing nothing(i.e. let $\tau_{in}\equiv0$). Now we consider the case when $x_1-\neq x-$ or $x_1+\neq x+$. In this case, we can find for each $n$ a positive real number $\tau_{1n}$ such that

$$\mathcal{A}(u_n(-\tau_{1n}))=\frac{1}{2}(\mathcal{A}(x-)+\mathcal{A}(x_1-))$$

Then when we consider the strip $\tau_{1n}*u_n$, we would obtain a convergent subsequence converging to some strip $v_1$ with $\mathcal{A}(v_1(0))=\frac{1}{2}(\mathcal{A}(x-)+\mathcal{A}(x_1-))$. We know that $\tau_{1n}>0$ for all $n$ since $\mathcal{A}(x_1-)\leq\mathcal{A}(x+)$, and by calculating the energy of $u_1$ in $[-M-\tau_{11},-\tau_{11}]\times[0,1]$ for $M>0$ sufficiently large and taking the limit on this subset, we obtain that the energy of $v$ on a compact subset $[-M,0]$ is close to $\frac{1}{2}(\mathcal{A}(x_1-)-\mathcal{A}(x-))$, so $\tau_{1n}$ is bounded above and we can then conclude that the left end of $v_1$ is the critical point $x-$. Similarly, one can show that the right end of $v_1$ will be $x_1-$. On the other hand, because of the conformal equivalence(and in fact the equivalence that preserves the energy) between strips $\mathbb{S}$ and the "punctured upper half-plane" $\mathbb{H}\setminus\{0\}$ and the following fact on energy estimate:

**3.47 Lemma.** *Assume $(X,J)$ is an almost complex manifold and $R$ a totally real submanifold of $X$, then there exists a positive integer $\delta>0$ such that for any chosen Riemannian metric $g$ and any pseudo-holomorphic map*

$$u\colon(\mathbb{H},\mathbb{R})\to(X,R),$$

*we have $\int_{\mathbb{H}}|\mathrm{d}u|^2\mathrm{dvol}_g\geq\delta$.*

With the existence of such a $\delta>0$, it follows directly that the sequences $\{\tau_{1n},\tau_{2n},\cdots\}$ should be finite, and the result follows from an induction process. $\square$

*Proof of Lemma 3.47.* This is just a simple Corollary of the mean value inequality A.1. This inequality gives a positive number $\delta > 0$, and if there is a $u$ with energy $E(u) < \delta$, then by mean-value inequality, we must have $|\mathrm{d}u| = \mathrm{O}(\frac{1}{r})$ for all $r > 0$, therefore we have $\mathrm{d}u = 0$ everywhere. $\square$

Theorem 3.46 gives the splitting of a pseudo-holomorphic strip under translations into a union of strips that connects through critical points, named "brocken trajectories". Note that if we have such a brocken trajectory, we could regard each critical point as a path and assign a canonical short path to each of these critical points. Then the whole brocken trajectories can be viewed as a decomposition of a compact Riemann surface, hence the Maslov-Viterbo index can be defined on this pair $(v_1, v_2, \cdots, v_k)$ as the sum of all of them.

**3.48 Proposition.** The index defined above coincides with the Maslov-Viterbo index of the original strip $u$.

Hence in the case when $\mathrm{Ind}(u) = 2$, it follows that $u$ splits into at most two trajectories $v_1$ and $v_2$, each with Maslov-Viterbo index 1, hence lying in $\mathcal{M}(x-, x)$ and $\mathcal{M}(x, x+)$, each with dimension 1. When we quotient the action of $\mathbb{R}$, we obtain that the there are only finitely many of them. There is still one thing we need to consider: we only prove that the boundary of $\mathcal{M}(x-, x+)$ lies in the product space $\mathcal{M}(x-, x) \times \mathcal{M}(x, x+)$, but we haven't shown that they coincide, so that by the classification of 1-manifolds the boundary points of a compact 1-manifold always appear in pairs and therefore proving the fact that $\partial^2 = 0$. This requires us to construct a sequence $\{u_n\} \subseteq \mathcal{M}(x_0, x_2)$ that tends to the pair $(v_1, v_2) \in \mathcal{M}(x_0, x_1) \times \mathcal{M}(x_1, x_2)$. In order to do this, we use a variation of Taubes' construction coming from his paper [Tau82]. The construction goes as follows: for $x_2 \in L_0 \cap L_1$, we consider an open neighbourhood $U$ of $x_2$ that is the image of a neighbourhood of 0 in $T_yP$ via the map $\exp_y \colon T_yP \times [0,1] \to P$ with $\exp_y(v, i) \in L_i$ when $v \in T_yL_i$. From the property of a pseudo-holomorphic strip mentioned in section 2, we know that there exists a positive number $M > 0$ such that for all $s \geq M$ we have

$$v_1(s,t) \subseteq U \quad \text{and} \quad v_2(-s,t) \subseteq U \quad \text{for all } t \in [0,1].$$

Hence for any such strip, there exists $\xi_i \in W^{1,p}(\mathbb{S}, T_yP)$ such that $\exp_y(t, \xi_i(s,t)) = v_i(s,t)$. Now pick an increasing smooth function $\beta \colon \mathbb{R} \to [0,1]$ such that $\beta(s) \equiv 0$ for $s \leq 0$ and $\beta(s) \equiv 1$ for $s \geq 1$ and construct

$$v_1 \# v_2(s,t) = \begin{cases} v_1(s+M+1, t), & s \leq -1; \\ \exp_y(t, \beta(-s)\xi_1(s+M+1,t) + \beta(s)\xi_2(s-M-1,t)), & -1 < s < 1; \\ v_2(s-M-1, t), & s \geq 1. \end{cases}$$

It is obvious that $v_1 \# v_2 \in \mathcal{P}^{1,p}(x_0, x_2)$. For this construction, our main conclusion is that

**3.49 Theorem.** Let $K_{01} \subseteq \mathcal{M}(x_0, x_1)$ and $K_{12} \subseteq \mathcal{M}(x_1, x_2)$ be compact subsets containing only **regular** trajectories, i.e. trajectories $u$ such that the linearization $\mathrm{d}\bar{\partial}_u$ is surjective(note that for a generic choice of $J$, this would be always the case as we have shown in the above transversality result), then there exists a positive number $\rho_0 > 0$ and a smooth map

$$\exp \colon K_{01} \times [\rho_0, +\infty] \times K_{12} \to \mathcal{M}(x_0, x_2)$$

mapping any pair $(u_1, s, u_2)$ to $\exp_{u_1\#u_2}(\xi)$, where $\xi$ is a vector field over $\mathbb{S}$ with the estimate $\|\xi\|_{W^{1,p}} \leq C\|\bar{\partial}(u_1\#u_2)\|_{L^p}$. Moreover, for any pair $(v_1, v_2) \in \mathcal{M}(x_0, x_1) \times \mathcal{M}(x_1, x_2)$ such that $\mathcal{M}(x_0, x_2) \cap U_\varepsilon(v_1, v_2)$ is contained in the image of exp. Here we define $U_\varepsilon(v_1, v_2)$ to be the union of loops $\gamma \in \Omega(P; L_0, L_1)$ such that $\mathrm{dist}(v_1(s), \gamma) < \varepsilon$ or $\mathrm{dist}(v_2(s), \gamma)$.

This tells us that the sequence $\exp_{v_1\#v_2}\xi$ is what we want, and therefore the boundary of $\mathcal{M}(x_0, x_2)$ is exactly $\mathcal{M}(x_0, x_1)\times\mathcal{M}(x_1, x_2)$. The proof of this theorem can be found in Floer's paper [Flo88c].

# 4. The Floer Cohomology

Now since we have constructed the Floer complex $(CF(P; L_0, L_1; J; \mathbb{Z}_2), \partial_F)$, we can take the cohomology to obtain the **Floer cohomology group** $HF(P; L_0, L_1; J; \mathbb{Z}_2)$. We must firstly show that this depends only on the manifold pair $(P; L_0, L_1)$ itself, that is, the cohomology group is independent of the choice of the generic $J$ and invariant under Hamiltonian isotopies $H$. Here for simplicity we just write $L = L_0$ and $L_1 = \phi_H(L)$. The main result is that

**4.1 Theorem** ([Flo88c])**.** Assume that $(J_1, H_1)$ and $(J_2, H_2)$ be two pairs of structures on $(P, \theta)$ such that $J_i$ is the $\theta$-compactible almost complex structure such that $L$ is totally real and that the linearization $\mathrm{d}\bar{\partial}_u$ is surjective for any pseudo-holomorphic strip $u$, and $H_i$ be a Hamiltonian such that the induced Hamiltonian diffeomorphism $\phi_H$ satisfies $\phi_H(L)$ intersects with $L$ transversely. Then we have the isomorphism

$$HF(P; L, J_1, H_1; \mathbb{Z}_2) \cong HF(P; L, J_2, H_2; \mathbb{Z}_2)$$

The proof can be seen in section 3 of this paper. This allows us to consider the case "$L_0 = L_1$", that is, $L_1$ is obtained from $L_0$ by an exact symplectic isotopy $\{\phi_t\}_{0\leq t\leq 1}$, then it is obvious that for $t$ sufficiently small, all the trajectories $u \in \mathcal{M}(x, y; L_0, \phi_t(L_0))$ has image in the tubular neighbourhood of $L$. This allows us to consider the finite-dimensional Morse theory in the cotangent bundle $T^*L$, with a given generating function $H$. Then we want to show that the Floer cohomology $HF(P, L; J, H; \mathbb{Z}_2)$ with $\mathbb{Z}_2$-coefficients is isomorphic to the Morse cohomology $H^*_M(L; \mathbb{Z}_2)$, hence proving the Arnold conjecture in case when $\pi_2(P, L) = 0$. To do this, we should relate the space of trajectories $\mathcal{M}_{\boldsymbol{J}}(x, y)$ for two intersection points $x, y$ when $J$ regular to the space of trajectories $\mathcal{L}(x, y)$ in the finite-dimensional Morse theory for $H$.

**4.2 Theorem.** Let $L$ be a smooth Riemannian manifold with Riemannian metric $g$. Let $H$ be a smooth function defined on $L$ such that there exists $\varepsilon > 0$ inserting into the estimate

$$|f(x)| + |\nabla f(x)| + |\nabla^2 f(x)| < \varepsilon.$$

Then there exists a family $\boldsymbol{J} = \{J_t\}_{0\leq t\leq 1}$ such that the natural projection

$$\mathcal{M}_{\boldsymbol{J}}(x, y) \to C^\infty(\mathbb{R}, L)\colon u \mapsto u_0(\tau) = u(\tau, 0)$$

where $x, y$ are critical points of $H$, gives a bijection of $\mathcal{M}_{\boldsymbol{J}}$ onto the set of all trajectories of the gradient flow of $H$ connecting $x$ to $y$.

Here we construct the Floer cohomology theory on the cotangent bundle $T^*L$. Moreover, let $\boldsymbol{J}$ be regular in the sense that the linearization is surjective, we would have

**4.3 Proposition.** For $u \in \mathcal{M}_{\boldsymbol{J}}(x, y)$, there is an isomorphism

$$\ker \mathrm{d}\bar{\partial}_u \cong T_p W^u(x) \cap T_p W^s(y)$$

on $p = u(0,0)$, where $W^u(x)$ is the unstable manifold of $x$ and $W^s(y)$ is the stable manifold of $y$, corresponding to the smooth function $H$.

And therefore we have the desired conclusion. The proof of both of these results can be found in Floer's paper [Flo89b].

# A. Cauchy-Riemann Equations

In this section we state and prove important facts about solutions of Cauchy-Riemann equations (1) with the given boundary condition that we will use in this paper. We will always assume $u$ is such a solution to equation (1).

**Aa) The Mean Value Inequality** We will firstly construct an analogue of mean value inequality of solutions to non-linear Cauchy-Riemann equations to the case on Laplace equations. For Laplacian case, this follows from a direct calculation. See [Eva10] for details of the proof. Here the case is slightly more complicated: for non-linear equations, we must firstly take the linearization and do the estimate for $\mathrm{d}u$, and when taking linearization, the non-linear term appears to make the equation much more complicated. However, we still have in some sense similar results to the easy linear case. In our case, the mean value inequality reads:

**A.1 Proposition** (Mean Value Inequality)**.** Let $(P, J)$ be a compact almost complex manifold and $L \subset M$ a totally real submanifold(i.e. $T_x L$ is a totally real subspace of $T_x P$ for all $x \in L$), then for a suitable choice of the Riemannian metric $g$, there exists constants $c, \delta > 0$ such that for every $r > 0$ and every holomorphic map $u \colon B_{2r}(0) \cap \mathbb{H} \to P$ with $u(B_{2r}(0) \cap \partial\mathbb{R}) \subset L$,

$$\int_{B_{2r}(0)\cap\mathbb{H}} |\mathrm{d}u|^2 < \delta \quad \Rightarrow \quad \sup_{B_r \cap \mathbb{H}} |\mathrm{d}u|^2 \leq \frac{2c}{r^2} \int_{B_{2r}\cap\mathbb{H}} |\mathrm{d}u|^2.$$

The idea of proof is to reduce the non-linear equation into some linear partial differential inequalities, and apply the mean value inequality for subharmonic functions to obtain the required estimate. First of all, with any Riemannian metric compactible with almost complex structure $J$, we have $|\mathrm{d}u|^2 = 2|\xi|^2 = 2|\eta|^2$ where $\xi = \dfrac{\partial u}{\partial s}$ and $\eta = \dfrac{\partial u}{\partial t}$. Pick $w = \dfrac{1}{2}|\mathrm{d}u|^2$ to be the energy density function and by a direct calculation, we would have the inequality

$$\Delta w \geq -c w^2. \tag{15}$$

and the problem is reduced to the analysis of the inequality (15). The calculation is non-trivial, and we need a choice of some proper Riemannian metric. This is done in U. Frauenfelder's master thesis [Fra08].

**A.2 Lemma** (Frauenfelder). *There exists a Riemannian metric $g$ defined on $(P, J, L)$ such that the following properties hold:*

1. *$g(Ju, Jv) = g(u, v)$ for all $u, v \in T_xP$ and all $x \in P$;*

2. *$JT_xL$ is orthogonal to $T_xL$ for all $x \in L$ w.r.t. $g$;*

3. *$L$ is a totally geodesic submanifold of $(P, g)$.*

*Proof.* Using partition of unity, it suffices to construct $g$ locally, i.e. in an Euclidean space $\mathbb{C}^n$, where $J$ is standard and $L$ is the subspace $\mathbb{R}^n \times \{0\} \subset \mathbb{C}^n$. The first and second property reads $g$ is a $2 \times 2$ block diagonal matrix such that

$$g = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

write $a_{ij}$ for the coefficients of the matrix $A$, and from the third property with some calculation, we know that the third property is equivalent to the fact that $\partial_{n+i}A(x, 0) = 0$ for all $x \in L$. It's obvious that under these requirements one Riemannian metric does exist locally near every point of $L$, but for $x \notin L$, the construction would be much easier since the second and third requirements are empty requirements. $\square$

Now we do the calculation. With notations at hand, we could compute the Laplacian of $w$, where $\Delta = \dfrac{\partial^2}{\partial s^2} + \dfrac{\partial^2}{\partial t^2}$ directly as

$$\frac{1}{2}\Delta w = |\nabla_s \xi|^2 + |\nabla_t \xi|^2 - g(R(\xi, \eta)\eta, \xi) + \kappa$$

where $R(\xi, \eta)$ is the curvature term $\nabla_s \nabla_t - \nabla_t \nabla_s$(Note that in this case, vector fields $\dfrac{\partial u}{\partial s}$ and $\dfrac{\partial u}{\partial t}$ commutes.) and $\kappa$ is an "error term" given by

$$\kappa = g(\nabla_s((\nabla_\eta J)\xi - (\nabla_\xi J)\eta), \xi) = g(\xi, (\nabla_\eta J)\nabla_s \xi - (\nabla_\xi J)\nabla_s \eta) + g((\nabla_s \nabla_\eta J)\xi - (\nabla_s \nabla_\xi J)\eta),$$

and since $P$ is compact and $u$ holomorphic, we have the estimate for $J$

$$|\nabla_s(\nabla_\eta J)|, |\nabla_s(\nabla_\xi J)| \le c(|\xi|^2 + |\nabla_t \xi|)$$

and the first order term controlled by $|\xi|^2$, thus we have

$$\kappa \ge -c_1|\xi|^4 - c_1|^2\xi|^2(|\nabla_s \xi| + |\nabla_t \xi|) \ge -\frac{1}{2}(|\nabla_s \xi|^2 + |\nabla_t \xi|^2) - \frac{c_1(1 + c_1)}{2}|\xi|^4$$

and hence

$$\frac{1}{2}\Delta w \ge \frac{1}{2}(|\nabla_s \xi|^2 + |\nabla_t \xi|^2) - \frac{c_1(1 + c_1)}{2}|\xi|^4 - g(R(\xi, \eta)\eta, \xi) \ge -c_2 w^2.$$

However, in our case there is a boundary in $B_{2r}(z_0) \cap \mathbb{R}$, so we must extend $w$ firstly to a function $\tilde{w}(s, t)$ such that when $t < 0$, we have $\tilde{w}(s, t) = w(s, -t)$. By some calculation of normal derivatives,

$$\frac{\partial w}{\partial t}(s, 0) = 2g(\nabla_t \xi, \xi)|_{t=0} = -2g(J\xi, \nabla_s \xi)|_{t=0} = 0,$$

thus we could extend $w$ through the boundary. Here we use the fact that $L \subset P$ is totally geodesic(then we will have $\nabla_s \xi$ is still orthogonal to $J\xi$). $\square$

Now we focus on this "subharmonic function" $w$. The result about the inquality $\Delta w \ge -cw^2$ for $c > 0$ is that

**A.3 Proposition.** Assume that $w\colon B_r(0) \to \mathbb{R}$ is $C^2$ and satisfies inequality (15), $w \geq 0$ and $\displaystyle\int_{B_r(0)} w < \frac{\pi}{8c}$, then

$$w(0) \leq \frac{8}{\pi r^2} \int_{B_r} w.$$

In our case, we firstly obtain the estimate for $\tilde{w}$, and note that the integral of $\tilde{w}$ in $B_{2r}(0)$ is identical to twice the integral of $w$ over $B_{2r}(0) \cap \mathbb{H}$, and that for every $z \in B_r(0) \cap \mathbb{H}$, we can pick a ball with center at $z$ of radius $r$ and apply proposition A.3 for every such $z$ to obtain our required result. □

*Proof.* We want to further simplify the problem to the case when $w$ is a positive subharmonic function, or say, a minus of such a function by some positive constant. In this case the mean value inequality just follows from the classical theory of Laplace equations. To do this, first of all, observe that the constant $c$ can be assumed to be 1 since we can do a rescaling, and the constant $r$ can also be assumed to be 1 since we can construct a new function $\tilde{w}$ by $\tilde{w}(s,t) = w(rs, rt)$, and we can verify that $\tilde{w}$ satisfies all the conditions with $r$ replaced by 1, and this construction gives a one-to-one correspondence between $w$ and $\tilde{w}$, so we can recover $w$ from $\tilde{w}$. The hardest part is to show that when we assume $c = 1$ and $r = 1$, we want to show $\Delta w \geq -b$ for some positive constant $b$. To do this, we need a "Heinz trick", work as follows: for $0 \leq \rho \leq 1$, consider the function

$$f(\rho) = (1-\rho)^2 \sup_{z \in B_\rho(0)} w(z),$$

from the construction we know that $f(1) = 0$ and $f$ is a non-negative continuous function, so there exists $\rho^* \in [0,1)$ such that $f(\rho^*)$ reaches the maximum of $f$, and write $a = w(z^*) = \sup_{z \in \rho^*} w(z)$, then if we pick $\varepsilon = \dfrac{1-\rho^*}{2}$, we have

$$\sup_{B_\varepsilon(z^*)} w(z) \leq \sup_{B_{\varepsilon+\rho^*}} w(z) = \frac{f(\rho^*+\varepsilon)}{(1-\rho^*-\varepsilon)^2} = \frac{4f(\rho^*+\varepsilon)}{(1-\rho^*)^2} \leq \frac{4f(\rho^*)}{(1-\rho^*)^2} = 4a,$$

thus in $B_\varepsilon(z^*)$ we have $\Delta w \geq -w^2 \geq -16a^2$, so $b = 16a^2$. Now from the lemma A.4, we know that

$$a = w(z^*) \leq \frac{16a^2r^2}{8} + \frac{1}{\pi r^2} \int_{B_r(z^*)} w = 2a^2r^2 + \frac{1}{\pi r^2} \int_{B_1} w$$

for all $r \leq \varepsilon$. Now we test if $4a\varepsilon^2 < 1$. If not, we can pick $r = \sqrt{\dfrac{1}{4a}}$ so that $r \leq \varepsilon$ and

$$\int_{B_1} w \geq \frac{\pi}{8}$$

contradicting the given condition. Therefore we must have $4a\varepsilon^2 < 1$ and if we pick $r = \varepsilon$, we have

$$a \leq 2a^2\varepsilon^2 + \frac{1}{\pi\varepsilon^2} \int_{B_1} w \leq \frac{a}{2} + \frac{4}{\pi(1-\rho^*)^2} \int_{B_1} w.$$

Therefore we have

$$w(0) = f(0) \leq f(\rho^*) = (1-\rho^*)^2 a \leq \frac{8}{\pi}\int_{B_1} w^2. \qquad \square$$

**A.4 Lemma.** *Assume that $w$ is a $C^2$-function, $w \geq 0$, $\int_{B_r(0)} w < \frac{\pi}{8c}$ and $\Delta w \geq -b$ for some positive constant $b$, then*

$$w(0) \leq \frac{br^2}{8} + \frac{1}{\pi r^2}\int_{B_r} w.$$

*Proof.* Let $v(s,t) = w(s,t) + \frac{b}{4}(s^2+t^2)$, then we must have $\Delta v = \Delta w + b$, thus $\Delta v \geq 0$ and, by the standard mean value property,

$$w(0) = v(0) \leq \frac{1}{\pi r^2}\int_{B_r} v = \frac{br^2}{8} + \frac{1}{\pi r^2}\int_{B_r} w. \qquad \square$$

**A.5 Remark.** *In Robbin and Salamon's paper [RS01] they treat with the more general case that the almost complex structure $J$ depends on the parameters $(s,t)$, and in this case we still have a weaker version of mean value inequality, which still fits in our proof of compactness.*

**Ab) Elliptic Regularity** In this paragraph we prove a regularity result for the solution to the Cauchy-Riemann equation (1) with the given boundary conditions. We firstly consider the linear case: assume that $\Omega \subset \mathbb{H}$ is a bounded open domain of the upper half-plane $\mathbb{H} = \{z | \mathrm{Im} z \geq 0\}$, and we consider the standard Euclidean space $\mathbb{C}^n$ with standard almost complex structure $J_0$. We write $(s,t)$ for the real and imagine variables of $\mathbb{H}$, and write $\bar\partial = \partial_s + J_0\partial_t$ for the anti-holomorphic differential. Consider the linear partial differential equation

$$\bar\partial u = 0, \tag{16}$$

where $u\colon \Omega \to \mathbb{R}^{2n}$ satisfies the boundary condition that $u(\mathbb{R}\times\{0\} \subset \mathbb{R}^n \times \{0\}$. We start with the regularity theorem for this special case.

**A.6 Proposition.** Assume $\Omega' \subset \Omega$ is an open subset of $\Omega$ such that $\overline{\Omega}' \subset \Omega$, then for any smooth function $\xi$ satisfying the boundary condition $\xi(\mathbb{R}\times\{0\}) \subset \mathbb{R}^n \times \{0\}$ defined on $\Omega$ and any positive integer $k$, there exists a constant $c_k > 0$ such that

$$\|\xi\|_{W^{k+1,2}(\Omega')} \leq c_k(\|\bar\partial\xi\|_{W^{k,2}(\Omega)} + \|\xi\|_{W^{k,2}(\Omega)}).$$

Note that when $\bar\partial\xi = 0$, this immediately gives the elliptic bootstrapping for the solution to equation (16).

*Proof.* Let $\langle -,- \rangle$ be the standard inner product in $\mathbb{R}^{2n}$. Assume that $\xi$ has compact support in $\Omega$, and by a direct calculation,

$$\begin{aligned}\int_\Omega |\bar\partial\xi|^2 &= \int_\Omega \langle \partial_s\xi + J_0\partial_t\xi, \partial_s\xi + J_0\partial_t\xi\rangle = \int_\Omega |\partial_s\xi|^2 + |\partial_t\xi|^2 + 2\langle\partial_s\xi, J_0\partial_t\xi\rangle \\ &= \int_\Omega |\partial_s\xi|^2 + |\partial_t\xi|^2 + \langle\partial_s\xi, J_0\partial_t\xi\rangle + \langle\partial_t\xi, J_0\partial_s\xi\rangle = \int_\Omega |\partial_s\xi|^2 + |\partial_t\xi|^2,\end{aligned}$$

thus by the Poincaré's inequality, we have $\|\xi\|_{W^{1,2}(\Omega')} \leq c_0 \|\bar{\partial}\xi\|_{L^2(\Omega)}$, and the other results follow directly from the elliptic bootstrapping. Now in general, we can pick a cut-off function $\beta\colon \Omega' \to \mathbb{R}$ such that $\beta(\Omega) = \{1\}$, then we can apply the previous regularity results to the function $\beta\xi$ and obtain the estimate

$$\|\beta\xi\|_{W^{k+1,2}(\Omega)} \leq c_k(\|\bar{\partial}(\beta\xi)\|_{W^{k,2}(\Omega)} + \|\xi\|_{W^{k,2}(\Omega)})$$

and observe that $\beta$ is a cut-off function on $\Omega$, with compact support, thus there is another positive constant, still written $c_k$, such that

$$\|\xi\|_{W^{k+1,2}(\Omega')} \leq \|\beta\xi\|_{W^{k+1,2}(\Omega)} \leq c_k(\|\bar{\partial}\xi\|_{W^{k,2}(\Omega)} + \|\xi\|_{W^{k,2}(\Omega)}). \qquad \square$$

We imagine this result as the first order estimate for the "first-order derivative" of a smooth map $u$, i.e. $\xi$ is some derivative of $u$, and this linear Cauchy-Riemann equation is just the linearized Cauchy-Riemann equation for the non-linear Cauchy-Riemann equation (1). To obtain an estimate for solutions to (1), we still need the regularity results for the standard Laplacian $\Delta$. Here $\Delta = \dfrac{\partial^2}{\partial s^2} + \dfrac{\partial^2}{\partial t^2}$. Here we are considering solutions with two type of boundary conditions on $\mathbb{R} \times \{0\}$: The Dirichlet boundary condition

$$u(s,0) = 0 \tag{17}$$

and the Neumann boundary condition

$$u_t(s,0) = 0 \tag{18}$$

for all $s \in \mathbb{R}$ such that $(s,0) \in \Omega$.

**A.7 Proposition.** Let $\Omega \subset \Omega''$ be bounded open subsets of $\mathbb{H}$ with $\bar{\Omega} \subset \Omega''$, then for every positive integer $k$, there exists a positive constant $c_k > 0$ such that

$$\|u\|_{W^{k+2,2}(\Omega)} \leq c_k(\|\Delta u\|_{W^{k,2}(\Omega'')} + \|u\|_{W^{k+1,2}(\Omega'')})$$

for every smooth function $u\colon \Omega'' \to \mathbb{R}$ that satisfies boundary conditions (17) or (18).

*Proof.* We consider the complex plane $\mathbb{R}^2 = \mathbb{C}$ with standard almost complex structure $J_0$, and write $\partial = \partial_s - J_0\partial_t$. Let $u, v$ be smooth functions on $\Omega''$ such that $v$ satisfies (17) and $u$ satisfies (18), then we set $\xi = (u,v)\colon \Omega'' \to \mathbb{R}^2$, and $\eta = \bar{\partial}\xi$, then $\xi$ and $\eta$ both satisfy the boundary condition in proposition A.6 and by this proposition, for each positive integer $k$ there exists positive constants $c_k > 0$ such that

$$\|\xi\|_{W^{k+2,2}(\Omega)} \leq c_k(\|\bar{\partial}\xi\|_{W^{k+1,2}(\Omega'')} + \|\xi\|_{W^{k+1,2}(\Omega'')})$$

then since $\bar{\partial}\xi = \eta$, using a similar elliptic estimate for $\partial$ to proposition A.6, we have

$$\|\xi\|_{W^{k+2,2}(\Omega)} \leq c_k(\|\Delta\xi\|_{W^{k,2}(\Omega'')} + \|\xi\|_{W^{k+1,2}(\Omega'')})$$

for another positive constant $c_k$. Therefore by taking components(that is, $u \equiv 0$ or $v \equiv 0$), we obtain the required estimate. $\square$

Now we go to our main estimate.

**A.8 Theorem.** Let $\Omega$ and $\Omega'$ be bounded open domains of $\mathbb{H}$ with $\overline{\Omega} \subset \Omega'$, and $u\colon \Omega' \to (M, J)$ be a holomorphic map with boundary conditions $u(\mathbb{R} \times \{0\}) \subset L$, where $L$ is a totally real submanifold of $M$, then for every positive constant $c_1 > 0$, there exists constants $c_k > 0$ for each positive integer $k$ such that

$$\sup_{\Omega'} |\partial_s u| \leq c_1 \quad \Rightarrow \|u\|_{C^k(\Omega)} \leq c_k.$$

*Proof.* Write $2n$ for the dimension of $M$, and for all $x \in L$, pick coordinate neighbourhoods $U_x$ such that $U_x \xrightarrow{\simeq} \mathbb{C}^n$ with $L \cap U_x$ identified with $\mathbb{R}^n \times \{0\}$. Then since $M$ is compact, we pick other covers that does not intersect $L$ and there would be finitely many such open neighbourhoods. Since $\Omega \subset\subset \Omega'$, there is a constant $\delta > 0$ such that for all $z \in \Omega$ we have $d(z, \partial\Omega' \setminus \mathbb{R} \times \{0\}) \geq \delta$, and hence the disk $H_\delta(z) = B_\delta(z) \cap \mathbb{H} \subset \Omega'$. From the first-order estimate $\sup |\partial_s u| \leq c_1$ we know that $d(u(z), u(z_0)) \leq c_1\delta$ for all $z \in H_\delta(z_0)$ with $z_0 \in \Omega$, hence for a small choice of $\delta$, we can assume that $u(H_\delta(z_0)) \subset U$ for some chosen open neighbourhood $U$ as above. Thus we could assume that $u\colon H_\delta(z_0) \to \mathbb{C}^n$ satisfies the boundary condition $u(\mathbb{R} \times \{0\}) \subset \mathbb{R}^n \times \{0\}$. If we write $u = (u_1, u_2)$ where $u_i$ is the $i$th component, then we have $\partial_t u_1(s, 0) = -\partial_s u_2(s, 0) = 0$ and hence from proposition A.7, we have

$$\|u\|_{W^{k+1,2}(H_{\delta/(k+1)}(z_0))} \leq c_k(\|\Delta u\|_{W^{k-1,2}(H_{\delta/k}(z_0))} + \|u\|_{W^{k+1,2}(H_{\delta/k})})$$

for some positive constant $c_k > 0$. Now since $\bar{\partial} u = 0$, we have

$$(\partial_s - J\partial_t)(\partial_s + J\partial_t)u = 0,$$

and hence

$$\Delta u = (\partial_t J)\partial_s u - (\partial_s J)\partial_t u.$$

This equality implies that

$$\|\Delta u\|_{W^{k-1,2}(H_{\delta/k})(z_0))} \leq c(\|\partial_s u\|_{W^{k-1,2}(H_{\delta/k})(z_0))} + \|\partial_t u\|_{W^{k-1,2}(H_{\delta/k})(z_0))}) \leq c\|u\|_{W^{k,2}(H_{\delta/k}(z_0))},$$

and therefore we would have

$$\|u\|_{W^{k+1,2}(H_{\delta/(k+1)}(z_0)} \leq c_k\|u\|_{W^{k+1,2}(H_{\delta/k}(z_0)}$$

Now the conclusion follows from the sobolev embedding theorem and the fact that there exists a positive constant $c > 0$ such that

$$\|u\|_{C^0(H_\delta(z_0)} \leq c\|u\|_{W^{2,2}(H_\delta(z_0))}. \qquad \square$$

For these theorems regarding Sobolev spaces, one can refer to [Eva10] for the explicit statements and proofs.

**Ac) The $L^p$-estimate** We need the $L^p$-estimate for all $p > 1$ for linear elliptic operators to serve the Fredholm theory of Cauchy-Riemann operators. Here by abuse of notation, we write the linearized Cauchy-Riemann operator as

$$L = \bar{\partial} + S\colon W^{1,p}(\mathbb{S}, \mathbb{R}^{2n}; L_0, L_1) \to L^p(\mathbb{S}, \mathbb{R}^{2n})$$

where $\bar{\partial}$ is the operator $\nabla_s + J_0\nabla_t$ with a chosen unitary frame for the almost complex structure $J(u)$, and $S$ is the order zero term such that $S(s,t)$ is bounded, converges to a bounded operator $S(t)$ as $s \to \infty$ and the first-order derivative $\nabla_s S(s,t)$ tends to 0 as $s \to \infty$. In this paragraph we show that for $L$ a similar estimate to the previous paragraph holds.

**A.9 Theorem.** For $p > 1$ and any $Y \in W^{1,p}(\mathbb{S}, \mathbb{R}^{2n}; L_0, L_1)$, there is a positive constant $C > 0$ such that

$$\|Y\|_{W^{1,p}(\mathbb{S})} \leq C(\|LY\|_{L^p(\mathbb{S})} + \|Y\|_{L^p(\mathbb{S})}).$$

Combining this result and the following local estimate

**A.10 Proposition.** Let $U \subset \mathbb{H}$ be a bounded open subset of the upper half-plane $\mathbb{H}$, $V \subset U$ an open subset such that $\bar{V} \subset U$, then for any $f \in L^p_{loc}(U)$ and $u \in L^p_{loc}(U)$ satisfying the equation $\bar{\partial}u = f$, we have $u \in W^{1,p}_{loc}(U)$ and there exists a positive constant $C > 0$ such that

$$\|u\|_{W^{1,p}(V)} \leq C(\|u\|_{L^p(U)} + \|f\|_{L^p(U)}).$$

The regularity for this linearized Cauchy-Riemann operator follows.

**A.11 Proposition.** Assume that $Y \in L^p(\mathbb{S}, \mathbb{R}^{2n}; L_0, L_1)$ satisfies $LY = 0$, then $Y \in W^{1,p}$ and therefore $Y$ is in the class $C^\infty$. Moreover, if $p > 2$, then $Y \in L^q$ for any $q > 1$.

Here $LY = 0$ holds in the distributional sense.

*Proof.* $LY = 0$ implies that $\bar{\partial}Y = SY$, and since $SY \in L^p(\mathbb{S})$ it follows from Proposition A.10 that $\bar{\partial}Y \in W^{1,p}_{loc}$ and by a bootstrapping procedure combined with the Sobolev embedding theorem, we have $\bar{\partial}Y \in C^\infty$ and therefore by applying estimates in theorem A.9, we deduce that $Y \in W^{1,p}(\mathbb{S})$.

The second part of this Proposition follows from the exponential decay property stated in Theorem 3.4. If $Y \in L^p$ for $p > 2$, then by Hölder's inequality it follows that

$$\int_0^1 \|Y(s,t)\|^2 \mathrm{d}t \leq \left(\int_0^1 \|Y(s,t)\|^p \mathrm{d}t\right)^{\frac{2}{p}}$$

since $Y$ lies in $L^p$, it follows that the energy of $Y$ does not diverge to $\infty$ as $s \to \infty$. Then $Y$ satisfies the exponential decay property and therefore $Y \in L^q$ for all $q > 1$. □

We now give a proof of Theorem A.11 and Proposition A.10. We start from Proposition A.10. To prove this local estimate, write $\partial = \nabla_s - J_0\nabla_t$ and observe that $\partial\bar{\partial} = \Delta$, so when $\bar{\partial}u = f$, it follows that $\Delta u = \partial f$ and therefore we can consider the following estimate for Laplacian operators:

**A.12 Proposition.** Assume that $u \in L^p_{loc}(U)$ satisfies

$$\Delta u = f + \nabla_s g + \nabla_t h$$

in the sense of distributions for $f, g, h \in L^p_{loc}(U)$, and $u$ satisfies either the Dirichlet or the Neumann boundary conditions, then $u \in W^{1,p}_{loc}(U)$ and for $V \subset\subset U$ there exists a positive constant $C > 0$ such that

$$\|u\|_{W^{1,p}(V)} \leq C(\|u\|_{L^p(U)} + \|f\|_{L^p(U)} + \|g\|_{L^p(U)} + \|h\|_{L^p(U)}).$$

*Proof.* Firstly assume $u$ to be harmonic, i.e. $\Delta u = 0$, then it follows from subsection Ae) that $u$ is smooth and satisfies the mean value property

$$u(x) = \frac{1}{\pi r^2} \int_{B(x,r)} u(y) \mathrm{d}y$$

in the extended region $\tilde{U} = U \cup \bar{U}$. Observe that the mean-value integral is identical to the convolution

$$\frac{1}{\pi r^2} \int_{B(x,r)} u(y) \mathrm{d}y = \chi_r * u(x)$$

where $\chi_r = \dfrac{1}{\pi r^2} 1_{B(0,r)}$. We then have

$$\chi_r * \chi_r(x) = \frac{1}{\pi^2 r^4} \int_{B(x,r)} \chi_r(y) \mathrm{d}y = \frac{1}{\pi^2 r^4} \left( 2 \arccos \frac{|x|}{2r} - \frac{|x|\sqrt{4r^2 - |x|^2}}{2} \right),$$

for those $x$ such that $B(x,r) \cap B(0,r) \neq \emptyset$ and is 0 otherwise. Thus $\chi_r * \chi_r \in C^0_0$. Then for any continuous function $h \in C^0$, we have the convolution

$$\pi r^2 (\chi_r * h)(s,t) = \int_{s-r}^{s+r} \left( \int_{t-\sqrt{r^2-(x-s)^2}}^{t+\sqrt{r^2-(x-s)^2}} h(x,y) \mathrm{d}y \right) \mathrm{d}x,$$

hence it follows that $\chi_r * g \in C^1$ and therefore the function $\psi = \chi_r * \chi_r * \chi_r$ is in $C^1_0$. By the identity $\psi * u = u$ we have

$$\|u\|_{W^{1,p}(\tilde{U})} = \|\psi * u\|_{W^{1,p}(\tilde{U})} \leq \|\psi\|_{C^1(\tilde{U})} \|u\|_{L^p(\tilde{U})}$$

and note that the value of $u$ in $U$ and $\bar{U}$ satisfies $u(s,-t) = u(s,t)$, thus we have

$$\|u\|_{W^{1,p}(U)} = \frac{1}{2^{1/p}} \|u\|_{W^{1,p}(\tilde{U})} \leq \frac{C}{2^{1/p}} \|u\|_{L^p(\tilde{U})} \leq C \|u\|_{L^p(\tilde{U})}.$$

In the general case, extend $u$ to the function $\tilde{u}$ defined in the region $\tilde{U}$, pick a bump function $\beta$ that is smooth, with support in $\tilde{U}$, and is identically 1 in $\tilde{V}$. Then we could consider the convolution

$$v = K * \beta f + K_1 * \beta g + K_2 * \beta h$$

where $K$ is defined in subsection Ae). Applying Calderón-Zygmund inequality, we have for any such $u$, there is a positive constant $C > 0$ such that

$$\| \operatorname{grad}(K_1 * \beta g)\|_{L^p(\tilde{U})} \leq C \|\beta g\|_{L^p(\tilde{U})} = C \|\beta g\|_{L^p(\operatorname{supp} \beta)}$$

and

$$\| \operatorname{grad}(K_2 * \beta g)\|_{L^p(\tilde{U})} \leq C \|\beta h\|_{L^p(\tilde{U})} = C \|\beta h\|_{L^p(\operatorname{supp} \beta)}.$$

For $f$, applying Young's inequality, we have

$$\| \operatorname{grad}(K * \beta f)\|_{L^p(\tilde{U})} \leq \|K_1 * \beta f\|_{L^p(\tilde{U})} + \|K_2 * \beta f\|_{L^p(\tilde{U})} \leq C \|f\|_{L^p(\operatorname{supp} \beta)}.$$

Combining with these three inequalities, we obtain the following estimate

$$\|\operatorname{grad} v\|_{L^p(\tilde{V})} \leq C(\|f\|_{L^p(\operatorname{supp}\beta)} + \|g\|_{L^p(\operatorname{supp}\beta)} + \|h\|_{L^p(\operatorname{supp}\beta)})$$

and applying the Poincaré's inequality, we have

$$\|\operatorname{grad} v\|_{W^{1,p}(\tilde{V})} \leq C(\|f\|_{L^p(\operatorname{supp}\beta)} + \|g\|_{L^p(\operatorname{supp}\beta)} + \|h\|_{L^p(\operatorname{supp}\beta)}).$$

Since the Laplacian of $v$ satisfies

$$\Delta v = \delta * \beta f + \delta_s * \beta g + \delta_t * \beta h = \beta f + \nabla_s(\beta g) + \nabla_t(\beta h),$$

so in $\tilde{V}$ we will have $\Delta(u - v) = 0$, i.e. $u - v$ is harmonic, and therefore we have the estimate

$$\|u\|_{W^{1,p}(\tilde{V})} \leq \|u-v\|_{W^{1,p}(\tilde{V})}+\|v\|_{W^{1,p}(\tilde{V})} \leq C(\|f\|_{L^p(\tilde{U})}+\|g\|_{L^p(\tilde{U})}+\|h\|_{L^p(\tilde{U})}+\|u\|_{L^p(\tilde{U})}).$$

□

Proposition A.10 follows immediately from this result.

*Proof of Proposition A.9.* Firstly, assume that $S$ is independent of $s$, i.e. we are considering the case on critical points, then we just write $S = S(t)$ and since $S$ is translationally invariant under $s$, we have for each $k \in \mathbb{Z}$,

$$\|Y\|_{W^{1,p}([k,k+1]\times[0,1])} \leq (\|LY\|_{L^p([k-1,k+2]\times[0,1])} + \|Y\|_{L^p([k-1,k+2]\times[0,1])}).$$

Summing them up with respect to $k$, we obtain the required global estimate. Note that here we are considering the region with two boundaries, and we just extend the function as usual, but consider the extended open subset to be properly contained in the extended region in order not to touch the boundary.

For the general case, observe that we are assuming $S$ converges to some operator $S_{\pm\infty}$ at the ends of the strip, hence is asymptotically independent of $s$. To be explicit, since $S(s,t) \xrightarrow{s\to\pm\infty} S_{\pm\infty}(t)$, for each $\varepsilon > 0$ there exists a positive integer $M > 0$ such that $\|S(s,t) - S_{+\infty}(t)\| < \varepsilon$ for any $s \geq M$, and similarly for any $s \leq -M$ we have $\|S(s,t) - S_{-\infty}(t)\| < \varepsilon$, therefore in the region $\mathbb{S}_M := ((-\infty, -M] \cup [M, +\infty)) \times [0,1]$ we have the estimate

$$\|Y\|_{W^{1,p}(\mathbb{S}_M)} \leq C(\|LY\|_{L^p(\mathbb{S}_M)} + \|Y\|_{L^p(\mathbb{S}_M)}),$$

and in any compact subset of $\mathbb{S}$, we have the local estimate A.10, hence we can pick a bump function $\beta$ such that $\beta \equiv 1$ on $[-M, M] \times [0,1]$ and is supported in $[-M-1, M+1] \times [0,1]$, and decompose $Y$ as $Y = \beta Y + (1-\beta)Y$, and we have the following estimate

$$\begin{aligned}
\|Y\|_{W^{1,p}(\mathbb{S})} &\leq \|\beta Y\|_{W^{1,p}([-M-1,M+1]\times[0,1])} + \|(1-\beta)Y\|_{W^{1,p}(\mathbb{S}_M)} \\
&\leq C(\|LY\|_{L^p([-M-1,M+1]\times[0,1])} + \|Y\|_{L^p([-M-1,M+1]\times[0,1])} + \|LY\|_{L^p(\mathbb{S}_M)} + \|Y\|_{L^p(\mathbb{S}_M)}) \\
&\leq C(\|LY\|_{L^p(\mathbb{S})} + \|Y\|_{L^p(\mathbb{S}_M)}).
\end{aligned}$$

□

**Ad) Unique Continuation** In this paragraph we will prove the unique continuation property for Cauchy-Riemann operators on closed unit disk $\mathbb{D}$. Floer, Hofer and Salamon proved in [FHS95] the transversality property for the moduli space of pseudo-holomorphic cylinders using this unique continuation property, and we will modify the proof of unique continuation in his paper and give a proof of the unique continuation property considering the pseudo-holomorphic disks. We start by a simple version of Riemann-Roch theorem. Assume $\Sigma$ is any compact Riemann surface(with boundary) and $E \to \Sigma$ is a complex vector bundle over $\Sigma$, then we define $\bar{\partial}\colon E \to \Omega^1(\Sigma) \otimes E$ as $f \mapsto \bar{\partial} f = \dfrac{\partial f}{\partial \bar{z}} \otimes \mathrm{d}\bar{z}$.

**A.13 Theorem** (Riemann-Roch)**.** Assume that $V$ is a complex vector space of dimension $r$, $L \subset V$ a totally real subspace and let $W^{1,p}(\mathbb{D}; V, L)$ be the space of all $W^{1,p}$-maps from $\mathbb{D}$ to $V$ such that $\partial\mathbb{D}$ is mapped into $L$, where $p > 2$ so that $W^{1,p} \subset C^0$. Then the Cauchy-Riemann operator

$$\bar{\partial}\colon W^{1,p}(\mathbb{D}; V, L) \to L^p(\mathbb{D}; \Omega^1(\mathbb{D}) \otimes V)$$

is Fredholm and its index

$$\mathrm{Ind}(\bar{\partial}) = \dim_{\mathbb{C}} V$$

as a complex operator.

We have proven that this operator is Fredholm in paragraph 3b), with an assumption that $L_1$ is Hamiltonian isotopic to $L_0$. Observe that when we write $\bar{\partial}$ in the form of differntial operator, it coincides with $\bar{\partial}$ as the standard outer differential on $\mathbb{D}$, hence we have

$$\mathrm{Ind}(\bar{\partial}) = \dim_{\mathbb{C}} \ker \bar{\partial} - \dim_{\mathbb{C}} \operatorname{coker} \bar{\partial} = r(h^{0,0} - h^{0,1})$$

where $V$ is regarded as the direct sum of $r$ trivial line bundles over $\mathbb{D}$ and $h$ is the Hodge number of $\mathbb{D}$. Since $\mathbb{D}$ is contractible, it follows that the index is just $r$. In order to prove the Riemann-Roch theorem, we still need to check that this operator is indeed surjective. As in [FHS95], we use the uniqueness of solutions to harmonic equations.

*Proof of theorem A.13.* Notice that solutions to the equation $\bar{\partial} = 0$ on $\mathbb{D}$ consisting of a pair $(u, v)$ of vector-valued functions satisfying $\bar{\partial} u = \bar{\partial} v = 0$ and on the boundary, $v \equiv 0$ and $\dfrac{\partial u}{\partial \nu} = 0$, here $\nu$ is the outer normal vector. Now it follows from the standard theory of existence and uniqueness of solutions to harmonic equations that $u \equiv 0$ and $v$ is constant in $\mathbb{D}$ with values in $L$. Hence $\dim_{\mathbb{C}} \ker \bar{\partial} = r$ and therefore from the index of $\bar{\partial}$ we have $\operatorname{coker} \bar{\partial} = 0$ and hence $\bar{\partial}$ is surjective. $\square$

Now let $S \in L^p(\mathbb{D}; \mathrm{End}(V))$ be a zero-order term and consider the perturbed Cauchy-Riemann operator $\bar{\partial} + S$, we have the following result generlized from [Vek62], named **Carleman's Similarity Principle**. Here we assume $J$ is dependent on the parameter $z \in U \subset \mathbb{C}$.

**A.14 Theorem** (Carleman's Similarity Principle)**.** Assume that $u \in W^{1,p}(H(z_0, \varepsilon), \mathbb{C}^n)$ is a solution to the non-linear Cauchy-Riemann equation

$$\partial_s u(z) + J(z)\partial_t u + S(z)u = 0 \tag{19}$$

where $H(z_0,\varepsilon)$ is a disk of radius $\varepsilon$ in $\mathbb{H}$, and $u$ satisfies the boundary condition that $u(H(z_0,\varepsilon)\cap\partial\mathbb{H})\subset\mathbb{R}^n\times\{0\}$, and we assume $J(z)\in W^{1,p}(H(z_0,\varepsilon),\mathrm{End}(\mathbb{C}^n))$ such that $\mathbb{R}^n\times\{0\}$ is always totally real and $u(z_0)=0$, then there exists a positive constant $\varepsilon\geq\delta>0$ and a family of endomorphisms $\Phi\in W^{1,p}(H(z_0,\delta),\mathrm{GL}_{\mathbb{R}}(\mathbb{C}^n))$ such that $J(z)\Phi(z)=\Phi(z)i$ the function $v(z)=\Phi(z)u(z)$ satisfies

$$\partial_s v(z)+J(z)\partial_t v(z)=0$$

for all $z\in H(z_0,\delta)$ and the boundary condition $v|_{H\cap\partial\mathbb{H}}\subset\mathbb{R}^n\times\{0\}$.

*Proof.* Pick a field $\Psi\in W^{1,p}(H(z_0,\varepsilon),\mathrm{GL}_{\mathbb{R}}(\mathbb{C}^n))$ of endomorphisms such that $\Psi(z)J(z)=\Psi(z)i$, and replace $u$ by $\Psi(z)\sigma(z)$, then we have $\sigma(z_0)=0$ and equation (19) becomes

$$\partial_s\sigma(z)+i\partial_t\sigma(z)+\tilde{S}(z)\sigma(z)=0$$

where $\tilde{S}(z)$ is a new field of endomorphisms such that $\tilde{S}(z)\in L^p$. Since $\Psi(z)$ commutes with $J(z)$, on $\partial\mathbb{H}$ we still have $\sigma(\partial\mathbb{H})\subset\mathbb{R}^n\times\{0\}$. Now we could decompose $\tilde{S}$ as complex linear part $\tilde{S}^+$ and complex anti-linear part $\tilde{S}^-$ so that $\tilde{S}=(\tilde{S}^++\tilde{S}^-)$, and we define a family $W(z)\in L^\infty(H(z_0,\varepsilon),\mathrm{End}_{\bar{\mathbb{C}}}(\mathbb{C}^n))$ of complex anti-linear operators such that $W(z)\sigma(z)=\sigma(z)$ for all $z\in H(z_0,\varepsilon)$. One example of such a family is given by setting $W(z)\zeta=\dfrac{\sigma(z)\sigma(z)^T\bar{\zeta}}{|\sigma(z)|^2}$ for all $z$ with $\sigma(z)\neq0$ and 0 for $\sigma(z)=0$. Then we could give a correction to $\tilde{S}$ by $U(z)=\tilde{S}^+(z)+\tilde{S}^-(z)W(z)$. By direct verification, $U(z)\sigma(z)=\tilde{S}(z)\sigma(z)$ for all $z\in H(z_0,\varepsilon)$ and since $W\in L^\infty$ and $\tilde{S}\in L^p$, it follows that $U(z)\in L^p$. For any $0<\delta<\varepsilon$, let $U_\delta=U(z)1_{H(z_0,\delta)}$ and for $\delta$ sufficiently small, we view $H(z_0,\delta)$ as part of the closed unit disk $\mathbb{D}$ via a biholomorphic map and consider the perturbed Cauchy-Riemann operator $D_\delta$ given by

$$D_\delta s=\bar{\partial}s+U_\delta s\mathrm{d}\bar{z}$$

for any $s\in W^{1,p}(\mathbb{D};V,L)$. Now we set $V=\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ since $\bar{\partial}$ is surjective, the operator $s\mapsto(\bar{\partial}s,s(0))$(or if $H(z_0,\delta)$ intersects with the boundary, $(\bar{\partial}s,s(1))$) is bijective, and since when $\delta\to0$, we have

$$\|U_\delta\|_{L^p}\xrightarrow{\delta\to0}0$$

hence the linear map $s\mapsto(D_\delta s,s(1))$ is bijective for $\delta$ sufficiently small(or when it does not touch the boundary, $s(0)$). Therefore we could find a section $s_\delta\in W^{1,p}(\mathbb{D},V)$ such that $D_\delta s_\delta=0$ with $s_\delta(0)=\mathrm{id}$ or $s_\delta(1)=\mathrm{id}$. In particular, on $H(z_0,\delta)$ we have $\partial_s s_\delta+i\partial_t s_\delta+Us_\delta=0$ and $s_\delta\xrightarrow{W^{1,p}}\mathrm{id}$ as $\delta\to0$. Set

$$\Phi(z)=\Psi(z)s(z),\quad v(z)=s(z)^{-1}\sigma(z)$$

then we have $\Phi(z)\in W^{1,p}$ and $\Phi(z)v(z)=\Psi(z)\sigma(z)=v(z)$. Moreover, in $H(z_0,\delta)$ we have

$$\begin{aligned}0&=\partial_s\sigma+i\partial_t\sigma+\tilde{S}\sigma=\partial_s(s(z)v(z))+i\partial_t(s(z)v(z))+U(z)s(z)v(z)\\&=(\partial_s s(z)+i\partial_t s(z)+U(z)s(z))v(z)+s(z)(\partial_s v(z)+i\partial_t v(z))=s(z)(\partial_s v(z)+i\partial_t v(z))\end{aligned}$$

and since $s(z)$ is invertible in $H(z_0,\delta)$, it follows that $\partial_s v(z)+i\partial_t v(z)=0$ in $H(z_0,\delta)$. □

With the Carleman's similarity principle at hand, we can prove several significant results. The first one is the isolatedness of critical points.

**A.15 Corollary.** Assume $u$ is a solution to (19) in $H(z_0, \varepsilon)$ with $J \in W^{l,p}$, $S \in W^{l-1,p}$ and $u(z_0) = 0$ satisfying the boundary conditions $u(\partial\mathbb{H}) \subset L$, then

1. There exists $0 < \delta < \varepsilon$ such that $u(z) \neq 0$ for all $z \in H(z_0, \delta) \setminus \{z_0\}$.
2. If $S = 0$, then there exists $0 < \delta < \varepsilon$ such that $\mathrm{d}u(z) \neq 0$ for all $z \in H(z_0, \delta) \setminus \{z_0\}$.

From elliptic regularity theorem A.11 we know that if $J \in W^{l,p}$ and $S \in W^{l-1,p}$ then already $u \in W^{l,p}$.

*Proof.* The first assertion follows directly by replacing $u$ by the corresponding pseudo-holomorphic map $v$ and apply the standard complex analysis results to $v$. For the second one, let $\xi = \partial_s u$ and differentiate (19), we have

$$\partial_s \xi + J\partial_t \xi - (\partial_s J)J\xi = 0$$

Apply similarity principle to $\xi$, we then obtain the required result. □

Our aim of this paragraph is to prove the following unique continuation property, which would be useful in the proof of transversality. Recall the following concept from differential topology [Hir76]:

**A.16 Definition.** Let $X, Y$ be differential manifolds, then the **k-jet** from $x \in X$ to $y \in Y$ is the equivalent class of all smooth functions $f \in C^\infty(U, Y)$ where $x \in U$ such that $f(x) = y$ and $f \sim g$ if and only if $(\mathrm{d}^i f)_x = (\mathrm{d}^i g)_x$ for all $0 \leq i \leq k$. When $k = \infty$, this means that $(\mathrm{d}^i f)_x = (\mathrm{d}^i g)_x$ for all $i \geq 0$.

This can be slightly generalized to the case of $W^{1,p}$-functions. A function $u \in W^{1,p}$ is called **vanishing at infinity order** at $x \in X$ if for any $k \geq 0$ there exists a positive constant $C_k > 0$ such that for all $r > 0$ sufficiently small and all $\varepsilon \in B(0, r)$, we have

$$\frac{|u(x+\varepsilon) - u(x)|}{r^k} \leq C_k.$$

Here we set $r$ sufficiently small such that $u$ takes values in a small neighbourhood of $x$. The set of all points such that the infinity jet of $f$ is zero(i.e. all its derivatives vanish) forms a closed subset of $X$, and in the case when $X, Y$ are complex manifolds and $f$ holomorphic, the local analycity of holomorphic functions(See, for example, [GH11]) gives that the set of points that a function vanishes at infinity order is also open, hence $f$ would be identically zero in the connected component of $X$ containing $x$. This is also called the "unique continuation property", in our case stated as follows:

**A.17 Theorem** (Unique Continuation)**.** Assume $J(z, u) \in W^{1,p}$ and $S(z, u) \in W^{1,p}$ depends on the map $u$. Let $u, v \colon \Omega \to \mathbb{C}^n$ be two $W^{1,p}$-solutions to equation

$$\partial_s u(z) + J(z, u)\partial_t u + S(z, u) = 0 \tag{20}$$

defined on some open subset $\Omega \subset \mathbb{H}$ satisfying $u(\partial\mathbb{H}) \subset \mathbb{C}^n$ and $v(\partial\mathbb{H}) \subset \mathbb{C}^n$. Then the set of points $z \in \Omega$ where $u - v$ vanishes to infinite order is open and closed. In other words, if $\Omega$ is connected, then $u \equiv v$ in $\Omega$.

*Proof.* Let $w = u - v$ and by direct calculation,

$$\begin{aligned}\partial_s w + J(z,u)\partial_t w &= -S(z,u) + S(z,v) + (J(z,v) - J(z,u))\partial_t v \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} S(z, u + s(u-v))\mathrm{d}s + \left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} J(z, u + s(v-u))\mathrm{d}s\right)\partial_t v \\ &= \tilde{S}(z)w\end{aligned}$$

where we assume $u$ and $v$ are fixed, then $w$ satisfies the linearized Cauchy-Riemann equation

$$\partial_t w + \tilde{J}(z)\partial_s w + \tilde{S}(z)w = 0$$

hence we could apply theorem A.14 to $w$, hence if $w$ vanishes at infinity order at some point $z \in \Omega$, then $w \equiv 0$ in $\Omega$. □

**Ae) Weyl's Lemma and Calderón-Zygmund Inequality** In this paragraph we will state the two important lemmas for use of elliptic regularity for Laplacian operator in [GT01]. This two lemmas are given without boundary conditions, but it is not hard to extend the result to our setting with Dirichlet and Nuemann boundary conditions. Recall that we have the mean value property for $C^2$-harmonic functions $u$, and a stronger result is that

**A.18 Proposition.** Assume that $u \in L^p_{loc}(U)$ where $U \subset \mathbb{C}$ is an open subset of $\mathbb{C}$ and $u$ satisfies the mean value property: for all $x \in U$ and $r > 0$ such that $B(x,r) \subset U$ we have the following equality

$$u(x) = \frac{1}{\pi r^2}\int_{B(x,r)} u(y)\mathrm{d}y,$$

then $u \in C^\infty(U)$ and $\Delta u = 0$.

With this result we can prove the converse, which is called Weyl's lemma:

**A.19 Corollary** (Weyl)**.** Assume that $u \in L^p_{loc}(U)$ satisfies $\Delta u = 0$ in the distributional sense, then $u$ satisfies the mean value property and hence $u \in C^\infty(U)$.

Now we assume that $u$ is a harmonic function in a bounded open subset $U \subset \mathbb{H}$ for $\mathbb{H}$ the upper half-plane satisfying either the Dirichlet boundary condition or the Neumann boundary condition, then we could construct an extension $\tilde{u}$ of $u$ to the open subset $U \cup \bar{U}$ where $\bar{U} = \{(s,-t)|(s,t) \in U\}$. Apply the continuation theorem in [Rud87] we know that $u$ is automatically harmonic in $U \cup \bar{U}$ and therefore harmonic and smooth on the boundary. Hence we have shown that

**A.20 Proposition.** Assume that $u \in L^p_{loc}(U)$ satisfies either the Dirichlet or Neumann boundary conditions with $U \subset \mathbb{H}$ and $\Delta u = 0$ in the distributional sense, then $u \in C^\infty(U)$.

Another important result is the Calderón-Zygmund inequality. Here we use the version stated in Mcduff and Salamon's book [MS04], which is easier in use.

**A.21 Proposition** (Calderón-Zygmund)**.** For every $p > 1$, there exists a positive constant $C > 0$ such that for every $C^\infty$-function $f$ with compact support in $\mathbb{R}^2$, we have the estimate

$$\|\operatorname{grad}(K_i * f)\|_{L^p} \leq C\|f\|_{L^p}.$$

*Proof.* In view of Calderón-Zygmund theory, it suffices for us to check this inequality for some $1 < p < \infty$. In fact, this inequality holds for $p = 2$. In order to see this, recall that on a Riemannian manifold $(M, g)$ we have the following Gauss' formula:

$$\int_N \mathrm{div} X \mathrm{d}\,\mathrm{vol}_g = \int_{\partial N} \langle \nu, X \rangle \mathrm{d}\,\mathrm{vol}_{\partial N},$$

and the Laplacian $\Delta$ is just the operator div grad, hence applying Gauss' formula with the estimate that for $u = K_i * f$, with the definition of $K_i$ and $f$, we have

$$|u(x)| + |\nabla u(x)| \leq \frac{C}{|x|^{n-1}}$$

for some positive constant $C$, hence we know that the integration $\displaystyle\int_{\partial B_R(0)} \left| u \frac{\partial u}{\partial \nu} \right| \mathrm{d}\,\mathrm{vol}_{\partial B_R(0)}$ vanishes as $R \to \infty$, therefore on $\mathbb{R}^n$, we have

$$\|\operatorname{grad} u\|_{L^2}^2 = \langle \operatorname{grad} u, \operatorname{grad} u \rangle_{L^2} = -\langle u, \Delta u \rangle_{L^2} = -\langle u, \partial_i f \rangle_{L^2} = \langle \partial_i u, f \rangle_{L^2} \leq \|\operatorname{grad} u\|_{L^2} \|f\|_{L^2}$$

therefore we have $\|\nabla u\|_{L^2} \leq \|f\|_{L^2}$. □

Here we write $K = \dfrac{1}{2\pi} \log |z|$ for the fundamental solution of $\Delta$, and $K_i$ the $i$th partial derivative of $K$. If we replace $\mathbb{R}^2$ by $\mathbb{H}$, similar estimates hold: we just extend functions to $\mathbb{R}^2$ and use the Calderón-Zygmund inequality.

# B. Pair of Pants Decomposition

This appendix serves as a proof of the existence of pair of pants decomposition for any Riemann surface $(\Sigma, g)$.

**B.1 Definition.** A **decomposition** of $\Sigma$ consists of two Riemann surfaces $\Sigma_1, \Sigma_2$ such that $\Sigma_1 \cup \Sigma_2 = \Sigma$ and $\Sigma_1 \cap \Sigma_2 = \partial\Sigma_1 \cap \partial\Sigma_2$.

We expect to decompose a compact Riemann surface into simple Riemann surfaces so that some properties that hold for simple Riemann surfaces would also hold for the complex one via induction process. The simplest but non-trivial Riemann surface with boundary is the following, called **a pair of pants** or **elementary cobordism** in the sense of Milnor [MSS65].

**B.2 Example.** *The Riemann surface $\Sigma$ that is obtained from the closed unit disk $\mathbb{D}$ by extracting two disjoint open disks in the boundary is called a pants. This is a compact Riemann surface with boundary $\coprod_{i=1}^{3} \mathbb{S}^1$. Another example is the closed unit disk $\bar{\mathbb{D}}$ itself, which is also a compact Riemann surface with only one boudary $\partial\mathbb{D} = \mathbb{S}^1$. A final example, and also a trivial one, is the cylinder* $\mathrm{C} = [0,1] \times \mathbb{S}^1$, *with boundary* $\partial\mathrm{C} = \mathbb{S}^1 \coprod \mathbb{S}^1$.

Now we could give the exact definition of "pair of pants decomposition".

**B.3 Definition.** A **pair of pants decomposition** for $\Sigma$ is an ascending sequence of Riemann surfaces

$$\Sigma_0 \subsetneq \Sigma_1 \subsetneq \Sigma_2 \subsetneq \cdots \subsetneq \Sigma_n = \Sigma$$

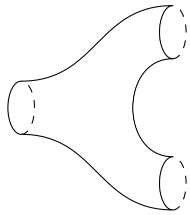

Figure 4: A Pair of Pants

such that $\Sigma_0$ is a closed disk, and $\Sigma_i$ admits a decomposition into $\Sigma_{i-1}$ and a pair of pants. When $i = n$, it's possible that $\Sigma_n$ is the union of $\Sigma_{n-1}$ and a closed disk instead of a pair of pants.

The picture of a pair of pants decomposition would be like as follows: here

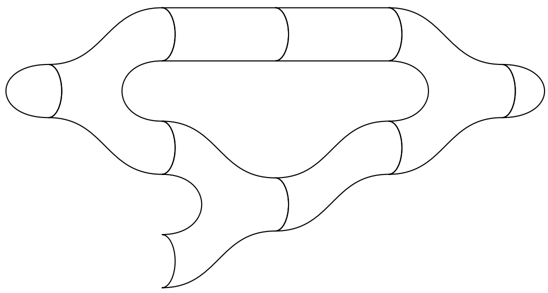

Figure 5: A Pair of Pants Decomposition

we add a pair of auxiliary cylinders in order to draw the picture beautifully. The main result of this section is that

**B.4 Theorem.** For any compact Riemann surface there exists a pair of pants decomposition.

We prove this theorem via Morse theory. We have stated some results in section 1, and here we need slightly more: the first one is that

**B.5 Proposition.** For any compact differential manifold $X$(possibly with boundary) the set $M(X)$ of all Morse functions on $X$ and the set $M(X; V, W)$ of all Morse functions on $X$ such that $f^{-1}(\min f) = V$, $f^{-1}(\max f) = W$ and the minimal and maximal values of $f$ are regular, where $V \cup W = \partial X$, $V, W$ are closed submanifolds without boundary of $\partial X$, are Baire sets in $C^\infty(\Sigma)$ endowed with both weak and strong topology.

The first existence can be seen in [Hir76], and the proof of the second one is only a slight modification of the first one, with the fact that all critical points of $f$ lies inside the interior of $X$.

**B.6 Definition.** A **gradient-like vector field** of $f$ on $X$ is a vector field $\xi$ such that

1. $\xi f(p) > 0$ for all $p \notin \mathrm{Crit}(f)$;
2. For any critical point $p \in \mathrm{Crit}(f)$, there exists an open neighbourhood $U$ of $p$ and a coordinate chart $g\colon U \to \mathbb{R}^\lambda \times \mathbb{R}^{n-\lambda}$ such that $f \circ g(x_1, x_2) = f(p) - |x_1|^2 + |x_2|^2$, where $\lambda = \mathrm{Ind}_f(p)$ is the Morse index of $f$.

The flow $\varphi_\xi^t$ of $\xi$ tends toward the direction that $f$ increases, and since $f$ cannot have critical values on the boundary, the gradient flow on the "incoming" boundary $V$ is pointing inward and on the "outgoing" boundary $W$ is pointing outward.

In order to do the pair of pants decomposition for any Riemann surface $\Sigma$, we should firstly determine $\Sigma_0$. Endow $\Sigma$ with a Riemannian metric $g$, pick any interior point $p \in \Sigma$ and consider a neighbourhood $B(p,r) := \{x \in \Sigma | d(x,p) < r\}$ isometric to $B(0,r) \subset \mathbb{C}$ with closure contained in the interior of $\Sigma$. We just set $\Sigma_0 = \overline{B(p,r)}$ and let $f_0 \colon \Sigma_0 \to$ to be the smooth function $f_0(x) := d(p,x)^2$. Expressed in coordinate charts, we can easily see that $f_0$ is a Morse function on $\Sigma_0$ with $p$ the unique critical point with Morse index 0. Now let $\Sigma_{01} = \Sigma \setminus \operatorname{Int} \Sigma_0$ to be the submanifold obtained by subtracting the interior of $\Sigma_0$ from $\Sigma$, and we obtain that $\partial\Sigma_{01} = \partial\Sigma_0 \cup \partial\Sigma$. For this Riemann surface, we can set $\partial\Sigma_0$ to be the "incoming boundary" and $\partial\Sigma$ the "outcoming boundary". Proposition B.5 gives the existence of a Morse function $f \colon \Sigma_{01} \to [a,b]$ with $f^{-1}(a) = \partial\Sigma_0$ and $f^{-1}(b) = \partial\Sigma$. Since the dimension of $\Sigma_{01}$ is two, it follows that the Morse index of critical points of $f$ can only be 1 if $f^{-1}(b) \neq \emptyset$ and can be two if $f^{-1}(b) = \emptyset$.(Here when $\partial\Sigma = \emptyset$ we just make $b$ to be slightly larger than the maximum of $f$ so that $f^{-1}(b) = \emptyset$). In order to see the decomposition for this Riemann surface $\Sigma_{01}$, let $\Sigma_x = f^{-1}(-\infty, x]$ for $x \in [a,b]$ and we see what happens to the family $\{\Sigma_x\}_{a \leq x \leq b}$ when $x$ passes through some critical value $c$ and some regular value $r$.

**B.7 Proposition.** Assume that there are no critical points in $f^{-1}([a,b])$, then $f^{-1}([a,b])$ is diffeomorphic to $f^{-1}(a) \times [0,1]$ and $f^{-1}(b)$ is diffeomorphic to $f^{-1}(a)$.

*Proof.* Since there are no critical points of $f$ in $f^{-1}([a,b]) := W$, the gradient-like vector field $\xi_f$ of $f$ is nowhere vanishing on $W$, hence we could normalize $\xi_f$ to be $\eta = \xi_f/\xi_f(f)$, so that $\eta(f) \equiv 1$, then the integral curve $\gamma$ of $\eta$ with any initial value $v \in f^{-1}(a) := V$ would satisfy

$$f \circ \gamma(t) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(f \circ \gamma(s))\mathrm{d}s = \int_0^t \eta(f)(s)\mathrm{d}s = t$$

hence for any $v \in V$ and $t \in [0, b-a]$, $\gamma_v(t)$ lies in $f^{-1}(t)$ and therefore the map $\gamma_*(*) \colon V \times [0, b-a] \to W$ gives a well-defined injective smooth map. That this is bijective comes from the existence of integral curves to the ordinary differential equation $\dot\gamma = \eta$ with initial value at any point $w \in W$, and consider the inverse flow of $\gamma_*(*)$. Then by closed map lemma and the fact that $\mathrm{d}\gamma_*(*)$ non-degenerate at every point of $V \times [0, b-a]$, $\gamma_*(*)$ is a diffeomorphism. $\square$

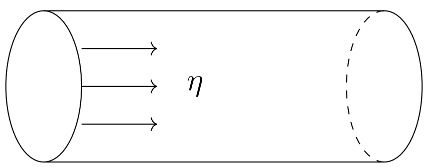


Figure 6: The Cylinder and Gradient-like vector field

Therefore when $\xi_f$ does not pass through any critical point, then the **sublevel set** $\Sigma_x$ is diffeomorphic to the product of a closed inteval and a boundary. When $\xi_f$ passes through a critical point, things become different. Roughly speaking, when $\xi_f$ passes through a critical point $p$, then $\Sigma_{f(p)+\varepsilon}$ for any $\varepsilon > 0$ would be obtained from $\Sigma_{f(p)-\varepsilon}$ by "attaching handles", and the number of handles are determined by the number of critical points on $f^{-1}(f(p))$. We firstly analyse the case when there is only one critical point lying in the fibre $f^{-1}(\frac{1}{2})$ in the submanifold $W = f^{-1}([0,1])$. Before stating the result, let's review some basic topological operations.

**B.8 Definition.** Assume that $M^m$ is an arbitrary differential manifold and $\varphi\colon \mathbb{S}^{\lambda-1}\times \mathbb{D}^{n-\lambda} \to M$ is an embedding, then a **surgery** of type $(\lambda, n-\lambda)$ for $\varphi$ is the quotient manifold $S(M,\varphi)$ of

$$(M \setminus \varphi(\mathbb{S}^{\lambda-1}\times 0)\coprod \mathbb{D}^{\lambda}\times \mathbb{S}^{n-\lambda-1}$$

given by equivalence relationship $\varphi(u,\theta v) \simeq (\theta u, v)$ where $u \in \mathbb{S}^{\lambda-1}$, $v \in \mathbb{S}^{n-\lambda-1}$ and $\theta \in (0,1]$.

The most intuitive example of a surgery would be a surgery of type $(1,2)$: in this case, the image of $\varphi$ is just a disjoint union of two closed disks and the surgery of type $(1,2)$ would be given by cancelling out the two centers of these two disks and attach a cylinder $\mathbb{S}^1 \times [0,1]$ via the rule given above. Then the resulting manifold $S(M,\varphi)$ would be a manifold given by subtracting the interiors of these two disks and attach the two ends of this cylinder to boundaries of these two disks. This is the so-called "attaching a handle to $M$". We also say a surgery of type $(r, n-r)$ is to attach an $r$-handle to the manifold $M$.

**B.9 Proposition.** Assume that $f$ is a Morse function defined on $f^{-1}([-\frac{1}{2},\frac{1}{2}]) = W$ with only one critical point with index $\lambda$ in $f^{-1}(0)$, then $f^{-1}(\frac{1}{2}) = S(V,f)$ is given by attaching a $\lambda$-handle to $V$. Then $W$ is a cobordism from $V$ to $S(V,f)$.

*Proof.* Here we only state the main idea of the proof. A complete proof can be found in chapter 3 of Milnor's lecture note [MSS65]. Near the critical point $p$, there is a Morse chart $\varphi\colon U \to B(0,\varepsilon)$ where $U$ is an open neighbourhood of $p$ and on $B(0,\varepsilon)$ the Morse function $f$ can be represented by

$$f\circ\varphi^{-1}(x) = -|x_1|^2 - \cdots - |x_\lambda|^2 + |x_{\lambda+1}|^2 + \cdots + |x_n|^2$$

hence we can draw a picture for the level sets in $B(0,2\varepsilon)$ as Where $V_{-\varepsilon}$ is the

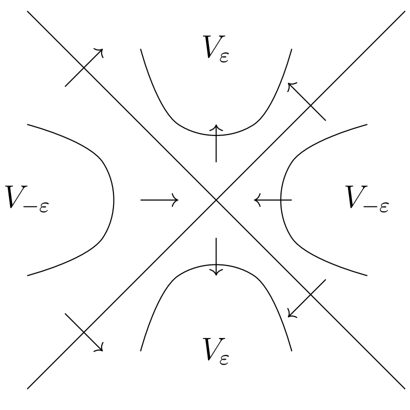


Figure 7: Local Model for Surgery

fibre $f^{-1}(-\varepsilon^2)$ and $V_\varepsilon$ the fibre $f^{-1}(\varepsilon^2)$. The directions of the gradient-like flow is indicated in this graph, hence we know that by cancelling the coordinate axis $\mathbb{R}^\lambda \times \{0\} \cup \{0\} \times \mathbb{R}^{n-\lambda}$, the grdient-like flow gives a diffeomorphism $V_{-\varepsilon} \setminus \mathbb{R}^\lambda \times \{0\} \to V_\varepsilon \setminus \{0\} \times \mathbb{R}^{n-\lambda}$, and when we pick a small tube-like closed neighbourhood of $\mathbb{R}^\lambda \times \{0\}$, the intersection of this neighbourhood with $V_{-\varepsilon}$ would be diffeomorphic to $\mathbb{S}^{\lambda-1} \times \mathbb{D}^{n-\lambda}$, and this intersection is sent via the gradient-like flow to $\mathbb{D}^\lambda \times \mathbb{S}^{n-\lambda-1}$ except at the center, which is exactly what we claim to be a surgery of type $(\lambda, n-\lambda)$. □

Notice that the attachment of handles is a local construction, i.e. we can choose the neighbourhood to be sufficiently small so that there would be only one such critical points in this neighbourhood, hence the argument can be easily generalized to any Morse functions, i.e. when a Morse function $f\colon (W; V_0, V_1) \to [0,1]$ has several critical points $\{p_1, \cdots, p_n\}$, then $V_1$ would be obtained from $V_0$ by attaching $n$ handles of several dimensions, determined by the Morse index of each critical point, and the construction can then be done disjointly, i.e. in our case, we have the following

**B.10 Proposition.** Any Riemann surface $\Sigma$ can be decomposed into a composition of several elementary cobordisms.

Given two differntial manifolds $V_0$ and $V_1$, we say the pair $(W; V_0, V_1)$ with $W$ a differential manifold a **cobordism** from $V_0$ to $V_1$ if $\partial W = V_0 \coprod V_1$ with the compatible orientation induced from $W$. A **composition** of cobordisms $(V_{01}; V_0, V_1)$ and $(V_{12}; V_1, V_2)$ is the cobordism $(V_{02}; V_0, V_2)$ given by glueing $V_{01}$ and $V_{12}$ via the identity map $V_1 \xrightarrow{\mathrm{id}} V_1$.

The final step of our proof of theorem B.4 is to show that the elementary cobordism in dimension 2 is, if connected, diffeomorphic to the pair of pants described above. The detailed proof is given in chapter 9 of Hirsch's book [Hir76], and here we only describe the ideas briefly.

**B.11 Proposition.** The elementary cobordism $(V_{01}; V_0, V_1)$ of dimension 2 is diffeomorphic to pair of pants if $V_{01}$ is connnected.

*Proof.* This relies on the fact that a surface $X$ such that there exists a Morse function $f\colon X \to [0,1]$ having three critical points with index $0, 0, 1$, then $X$ is diffeomoprhic to $\mathbb{D}^2$. The proof of this can be found in section 9.2 of Hirsch [Hir76], is quite technical, and hence we omit here. Since $V_0$ and $V_1$ are closed 1-manifolds, they must be disjoint unions of circles, and since $V_{01}$ is connected and orientable, we must have $V_0 \cong \mathbb{S}^1$ and $V_1 \cong \mathbb{S}^1 \coprod \mathbb{S}^1$ or conversely. Now assume the first case, and we attach two closed disks to $V_2$ to obtain $V_2'$ with prescribed Morse function with only a critical point of index 0 at the origin, and consider the union of $V_{01}$ and $V_2'$ along the boundary $V_2$ with a glued Morse function on the union, then this Morse function would have two critical points of index 0 and one critical points of index 1, and is therefore diffeomorphic to the disk $\mathbb{D}^2$. Therefore $V_{01}$ is obtained from $\mathbb{D}^2$ by subtracting two open disks, hence is diffeomorphic to the pair of pants. □

And theorem B.4 is proved. □

# C. Methods of Harmonic Analysis

In this section we treat with methods in harmonic analysis involved in this paper. This is mainly subtracted from Simon's book [Sim15].

**Ca) Calderón-Zygmund Method** This method is provided by Calderón and Zygmund [CZ52] exactly when they are deducing the Calderón-Zygmund inequality for Laplacian operators(or to say, for the operator $T\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ which is given by convolution by $K$). We will introduce the more general theory concerning such operators of convolution type, where the kernel is some function with behaviour very similar to this convolution by fundamental solutions $K_j$. This is also a special case of the so-called **Riesz potential**.

We start with a description of such a kernel.

**C.1 Definition.** We give the following assumptions on a funcction $K$ on $\mathbb{R}^n$:

$$|K(x)| \leq A|x|^{-n}; \tag{21}$$

$$K \text{ is } C^1 \text{ on } \mathbb{R}^n \setminus \{0\} \text{ and } |\nabla K(x)| \leq B|x|^{-n-1}; \tag{22}$$

and for each $0 < r < \infty$,

$$\int_{\mathbb{S}^{n-1}} K(r\omega)\mathrm{d}S(\omega) = 0 \tag{23}$$

where $\mathrm{d}S$ is the normalized rotation-invariant measure on the sphere $\mathbb{S}^{n-1}$.

A $C^1$-function $K$ on $\mathbb{R}^n \setminus \{0\}$ satisfying (23) and $K(\lambda x) = \lambda^{-n}K(x)$ for all $\lambda > 0$ and all $x \in \mathbb{R}^n \setminus \{0\}$ is called a **classical Calderón-Zygmund kernel**, and the corresponding linear map $T$ is called the **classical Calderón-Zygmund operator**.

The main result concerning such a function $K$ is that

**C.2 Theorem.** Let $K$ be a function defined on $\mathbb{R}^n \setminus \{0\}$ satisfying conditions (21) and (22), and the corresponding linear map $T$ maps $L^r(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$ for some $1 < r < \infty$, then for all $1 < p < \infty$, $T$ is a bounded linear map from $L^p$ to $L^p$.

We apply interpolation to prove this theorem. That is, it suffices for us to prove that $T$ is at least of weak type $(1, 1)$. That is, if $\mu$ is a measure on $\mathbb{R}^n$, then we have

$$\mu\{x \in \mathbb{R}^n ||Tf(x)| > \alpha\} \leq \frac{C}{\alpha}\|f\|_1$$

for any $\alpha > 0$, any $f \in L^1$ and some positive constant $C > 0$. Marcinkiewicz interpolation then tells us for any $1 < p < r$ we have $T$ is of strong type $(p, p)$. Using a dual argument(that is, consider the dual space of $L^r$ and replacing $T$ by the dual operator $T^*$, so the $L^r$-boundedness of $T$ implies $L^{r'}$-boundedness of $T^*$, which gives the $L^p$-boundedness of $T$ for $p > r$. Then we pick one $p > r$ and repeat the argument to conclude the proof). The rest of this subsection deals with the weak $L^1$-boundedness.

The proof of this uses the Calderón-Zygmund decomposition, whose idea is roughly given as follows: we decompose an $L^1$ function $f$ into two different parts $f = b + g$, where $b$ stands for "bad" and $g$ for "good". The reason why we call $g$ good is that its essential upper bound is finite and its $L^1$-norm is equal to the $L^1$-norm of $f$. Rigorously, the decomposition is stated as follows:

**C.3 Theorem** (Calderón-Zygmund). Given a non-negative function $f \in L^1(\mathbb{R}^n)$ and $\alpha \in (0, \infty)$, there is a disjoint family $\{Q_j\}_{j=1}^J$($J$ finite or infinite) of standard dyadic cubes(perhaps of different sizes) and measurable functions $b$ and $g$(with $g \geq 0$), so that

1. $f = b + g$;
2. $\sum_j |Q_j| \leq \alpha^{-1}\|f\|_1$; (24)
3. On $\mathbb{R}^n \setminus \bigcup_{j=1}^J Q_j$ we have
$$f(x) = g(x) \in [0, \alpha], \quad \forall x \notin \bigcup_{j=1}^J Q_j;$$
4. $g(x) = g_j$ on $Q_j$ where
$$g_j = \frac{1}{|Q_j|}\int_{Q_j} f(x)\mathrm{d}x$$
is a constant;
5. $\|g\|_\infty \leq 2^n\alpha$; (25)
6. If $b_j = b|_{Q_j}$, then we have
$$\int_{Q_j} b_j(x)\mathrm{d}x = 0;$$
7. $\|g\|_1 = \|f\|_1$ and $\|b\|_1 \leq 2\|f\|_1$;
8. $\|b_j\|_1 \leq 2^{n+1}\alpha|Q_j|$;
9. For any $r \in [1, \infty)$ we have
$$\frac{\|g\|_r^r}{\alpha^r} \leq 2^{n(r-1)}\frac{\|f\|_1}{\alpha}.$$

*Proof.* The most non-trivial part is the construction of $b$ and $g$, and the other properties will be easily verified. Take a dyadic decomposition of $\mathbb{R}^n$ by cubes, and let $k$ be the smallest integer such that

$$2^{-nk}\int f(y)\mathrm{d}y < \alpha.$$

This means that for any cube $Q$ of radius $2^k$, we have

$$\frac{1}{|Q_j|}\int_{Q_j} f(y)\mathrm{d}y < \alpha.$$

Therefore it is possible for us to choose cubes of radius $2^{-nk-1}$ that violates the inequality given above. Choose these cubes as $Q_j$, and since we have

$$|Q_j| \leq \frac{1}{\alpha}\int f(y)\mathrm{d}y,$$

there are only finitely many of them. Then we just iterate the procedure and subtracted from the class of dyadic cubes more such cubes, and we write these cubes as a collection $\{Q_j\}$. By construction, the inequality (24) is trivially satisfied, so is the bound (25). Here we just set $g = g_j$ on each $Q_j$ and $g = f$ outside the union, and outside these cubes, we have

$$\frac{1}{|Q|}\int_Q f(y)\mathrm{d}y < \alpha$$

holds for any cube $Q$, then from an analogue of Lebesgue differentiation theorem(in fact, it can be done by looking at the maximal functions), we obtain that $f(x) \leq \alpha$ for a.e. $x \notin \bigcup_j Q_j$. The construction of $g$ tells us $\|g\|_1 = \|f\|_1$, and since $b = f - g$, it follows that $\|b\|_1 \leq 2\|f\|_1$. Combining the essential bound of $g$ and the fact that $g|_{Q_j} = g_j$, it follows that $\|b_j|_1 \leq 2^{n+1}\alpha|Q_j|$. Finally, we are left with the comparison between $L^r$-norms of $g$ and $L^1$-norms of $f$. This follows from the essential bound of $g$. In fact, we have

$$\frac{\|g\|_r^r}{\alpha^r} \leq \frac{\|g^{r-1}\|_\infty \|g\|_1}{\alpha^r} \leq 2^{(r-1)n}\frac{\|g\|_1}{\alpha} = 2^{n(r-1)}\frac{\|f\|_1}{\alpha}. \qquad \square$$

Using this theorem, we canary directly proof the weak $L^1$-bound of $T$.

**C.4 Theorem.** Let $T$ be an operator of strong type $(r, r)$ for some $r$ and is given by a kernel $K$ satisfying conditions (21) and (22). Then there exists a positive constant $C > 0$ such that for all $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\mu\{x||Tf(x)| \geq \alpha\} \leq \frac{C}{\alpha}\|f\|_1.$$

In other words, $T$ is of weak type $(1, 1)$.

*Proof.* Without loss of generality, we can assume that $f$ is non-negative, and we apply the Calderón-Zygmund decomposition to $f$ to obtain $f = b+g$ where $b$ and $g$ satisfies conditions given in theorem C.3. Since $T$ is linear, we have $Tf = Tg+Tb$, hence we have

$$\mu\{x||Tf(x)| \geq \alpha\} \leq \mu\{x||Tg(x)| \geq \frac{\alpha}{2}\} + \mu\{x||Tb(x)| \geq \frac{\alpha}{2}\}.$$

For simplicity, we just write $d_g(\alpha)$ for the distribution function for $g$. Since $T$ is of strong type $(r, r)$, we already have

$$d_{Tg}(\frac{\alpha}{2}) \leq \frac{2^r}{\alpha^r}\int_{|Tg|\geq\frac{\alpha}{2}} |Tg(x)|^r \mathrm{d}x \leq \frac{2^r C^r}{\alpha^r}\|g\|_{L^r}^r \leq \frac{2^{nr-n+r}C^r}{\alpha}\|f\|_{L^1}.$$

Now it suffices to consider the second term $d_{Tb}(\frac{\alpha}{2})$. To do this, let $B_j$ be a ball with the same center as $Q_j$ but with radius $2R_j = 2\cdot 2^{n_j}(\frac{\sqrt{n}}{2})$, where the radius of $Q_j$ is $2^{n_j}$. Therefore $B_j$ is a ball with radius twice the smallest ball containing $Q_j$. Then we could decompose $d_{Tb}(\frac{\alpha}{2})$ into two parts:

$$d_{Tb}(\frac{\alpha}{2}) = \sum_j |B_j| + \mu\{x \notin \bigcup_j B_j||Tb(x)| \geq \frac{\alpha}{2}\},$$

and for the first part we have

$$\sum_j |B_j| = \sqrt{n^n}\tau_n \sum_j |Q_j| \leq \frac{C}{\alpha}\|f\|_1,$$

where $\tau_n$ is the volume of the unit sphere in $\mathbb{R}^n$. The second part is somewhat technical, here we have the conditions of $K$ involved. By Chebyshev inequality,

$$\mu\{x \notin \bigcup_j B_j ||Tb(x)| \geq \frac{\alpha}{2}\} \leq \frac{2}{\alpha}\int_{\mathbb{R}^n\setminus\bigcup B_j} |Tb(x)|\mathrm{d}x \leq \frac{2}{\alpha}\int_{\mathbb{R}^n\setminus B_j} |Tb(x)|\mathrm{d}x$$

for one $j \in J$. Let $x_j$ be the center of $Q_j$, and the integration is taken outside $B_{2R_j}(x_j)$. From the lemma

**C.5 Lemma.** *Let $h$ be an $L^1$-function supported in a ball $B_R(x_0)$ around some $x_0$ and suppose that $\int h(y)\mathrm{d}y = 0$. Then with $T$ given by the kernel $K$ satisfying conditions* (22)*, we have*

$$\int_{|x-x_0|\geq 2R} |Th(x)|\mathrm{d}x \leq 2^n C\omega_{n-1}\|h\|_{L^1}$$

*where $\omega_{n-1}$ is the area of $\mathbb{S}^{n-1}$, the unit sphere in $\mathbb{R}^n$.*

we readily know that

$$\int_{\mathbb{R}^n\setminus B_j} |Tb(x)|\mathrm{d}x \leq 2^{n+1}C\omega_{n-1}\alpha^{-1}\|f\|_1. \qquad \square$$

*Proof of Lemma C.5.* Without loss of generality, assume $x_0 = 0$. Since $\int h(y)\mathrm{d}y = 0$, we can write the left-hand side of the required inequality as

$$\int_{|x|\geq 2R}\left|\int_{\mathbb{R}^n}(K(x-y)-K(x))h(y)\mathrm{d}y\right|\mathrm{d}x \leq \int_{|x|\geq 2R}\int_{\mathbb{R}^n}|K(x-y)-K(x)||h(y)|\mathrm{d}y\mathrm{d}x.$$

Since $K$ satisfies condition (22), we have

$$|K(x-y)-K(x)| \leq \int_0^1 |\nabla K(x-\theta y)|\mathrm{d}\theta \leq \frac{B}{|x|^{n-1}},$$

and plugging in this inequality, we have

$$\text{LHS} \leq \int_{|x|\geq 2R}\int_{\mathbb{R}^n}\frac{B|h(y)|}{|x|^{n-1}}\mathrm{d}y\mathrm{d}x \leq 2^n\omega_{n-1}B\|h\|_{L^1}. \qquad \square$$

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